WL-TR-94-3030

DESIGN OF ROBUST OPTIMAL CONTROL SYSTEMS AND STABILITY ANALYSIS OF REAL STRUCTURED UNCERTAINTIES

BOR-CHIN CHANG
AJMAL YOUSUFF

DREXEL UNIVERSITY
PHILADELPHIA, PA 19104

MARCH 1994

FINAL REPORT FOR THE PERIOD AUGUST 1990 TO MARCH 1994

APPROVED FOR PUBLIC RELEASE; DISTRIBUTION UNLIMITED.

FLIGHT DYNAMICS DIRECTORATE
WRIGHT LABORATORY
AIR FORCE MATERIEL COMMAND
WRIGHT-PATTERSON AIR FORCE BASE, OHIO 45433-6553

NOTICE

When Government drawings, specifications, or other data are used for any purpose other than in connection with a definitely Government-related procurement, the United States Government incurs no responsibility or any obligation whatsoever. The fact that the government may have formulated or in any way supplied the said drawings, specifications, or other data, is not to be regarded by implication, or otherwise in any manner construed, as licensing the holder, or any other person or corporation; or as conveying any rights or permission to manufacture, use, or sell any patented invention that may in any way be related thereto.

This report is releasable to the National Technical Information Service (NTIS). At NTIS, it will be available to the general public, including foreign nations.

This report has been reviewed and is approved for publication.

Siva S. Banda

SIVA S. BANDA
Aerospace Engineer
Control Dynamics Branch

Frank R Swortzel

FRANK R. SWORTZEL
Chief, Control Dynamics Branch
Flight Control Division

David P. Lemaster

DAVID P. LEMASTER
Chief, Flight Control Division
Flight Dynamics Directorate

If your address has changed, if you wish to be removed from our mailing list, or if the addressee is no longer employed by your organization please notify WL/FIGC, WPAFB, OH 45433-7531 to help us maintain a current mailing list.

Copies of this report should not be returned unless return is required by security considerations, contractual obligations, or notice on a specific document.

REPORT DOCUMENTATION PAGE

Form Approved
OMB No. 0704-0188

1. AGENCY USE ONLY *(Leave blank)*

2. REPORT DATE
March 1994

3. REPORT TYPE AND DATES COVERED
Final-August 1990 to March 1994

4. TITLE AND SUBTITLE
Design of Robust Optimal Control Systems and Stability Analysis of Real Structured Uncertainties

5. FUNDING NUMBERS
PE-61102F
PR-2304
TA-N2
WU-08
F33615-90-C-3613

6. AUTHOR(S)
Bor-Chin Chang
Ajmal Yousuff

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)
Drexel University
Philadelphia, PA 19104

8. PERFORMING ORGANIZATION REPORT NUMBER
WL-TR-94-3030

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)
Flight Dynamics Directorate (WL/FIGC)
Wright Laboratory
Air Force Materiel Command
Wright-Patterson AFB, OH 45433-7533

10. SPONSORING/MONITORING AGENCY REPORT NUMBER

11. SUPPLEMENTARY NOTES

12a. DISTRIBUTION/AVAILABILITY STATEMENT
Approved for Public Release;
Distribution is Unlimited

12b. DISTRIBUTION CODE

13. ABSTRACT *(Maximum 200 words)*
This report addresses the computational problems arising in the design of robust controllers via H-inf optimization and mu-synthesis and proposes several new controller reduction approaches. Through our discovery of the convexity and monotonicity properties of the general parameter dependent H-inf Riccati equation, a quadratically convergent algorithm which is faster than the other existing search schemes is developed to compute the optimal H-inf norm. It is desired to preserve the closed-loop stability and performance while reducing the order of the controller. A structured truncation technique based on the closed-loop controllability and observability gramians, an H-inf controller parameterization and reduction method, and an observer-based controller parameterization and reduction procedure are employed to achieve this objective.

14. SUBJECT TERMS
Robust control, H-inf optimization, Mu-synthesis, Riccati equation, Controller reduction

15. NUMBER OF PAGES
148

16. PRICE CODE

17. SECURITY CLASSIFICATION OF REPORT	18. SECURITY CLASSIFICATION OF THIS PAGE	19. SECURITY CLASSIFICATION OF ABSTRACT	20. LIMITATION OF ABSTRACT
UNCLASSIFIED	UNCLASSIFIED	UNCLASSIFIED	DTIC USERS

NSN 7540-01-280-5500

Standard Form 298 (Rev. 2-89)
Prescribed by ANSI Std Z39-18
298-102

FOREWORD

This report was prepared by the Department of Mechanical Engineering and Mechanics, Drexel University, Philadelphia, Pennsylvania, under the Air Force Contract F33615-90-C-3600 sponsored by AFOSR. The work was performed under the direction of Drs. S. S. Banda and H. H. Yeh of the Flight Dynamics Laboratory, Wright Research & Development Center, Wright-Patterson Air Force Base, Ohio.

The technical work was conducted by Dr. B. C. Chang, Principal Investigator, Dr. A. Yousuff, Co-Principal Investigator, and their research assistants O. Ekdal, H. Bouguerra, X. P. Li, R. Zong, J.C. Chen, M. Breida, and I. Dias during the period August 1990 - March 1994.

The investigators in this study would like to thank to Dr. S. S. Banda and Dr. H. H. Yeh for their continuous encouragement and numerous valuable technical discussions.

Table of Contents

Table of Contents (Cont'd)

CHAPTER 1

INTRODUCTION

This document is the final report on the AFOSR supported research, "Design of Robust Optimal Control Systems and Stability Analysis of Real Structured Uncertainties," (AFOSR Contract F33615-90-C-3613). The report summarizes our research progress in the following problems: design of robust controllers via H^∞ optimization, investigation of the properties of the parameter-dependent H^∞ Riccati equation, fast computation of the optimal H^∞ norm, design of robust controllers via μ-Synthesis, controller reduction by structured trunction, a parametrization approach to reduced-order H^∞ controller design, controller reduction via observer-based controller parametrization, and dominant modes of mechanical systems.

H^∞ optimization theory plays a key role in many robust control problems such as tracking and disturbance attenuation, mixed sensitivity minimization, model matching, and μ-synthesis [1-10]. One of the most elegant and efficient solutions to the H^∞ optimization problem is the state space approach proposed by Doyle, Glover, Khargonekar, and Francis [7] and Glover and Doyle [8]. As shown in [7,8], the major computation involved in this approach is solving two γ-dependent H^∞ Riccati equations. The parameter γ serves as an upper bound of the H^∞ norm of the closed loop system. If, for a given γ, the two H^∞ Riccati equations have positive semidefinite stabilizing solutions and the spectral radius of the product of the two solutions is less than γ^2, then all stabilizing controllers can be easily constructed from the plant data and the Riccati solutions. The dimension of the resulting controller is not higher than that of the generalized plant and the H^∞ norm of the closed loop system is less than γ.

In Chapter 2, we will discuss how to formulate a robust control problem as an H^∞ optimization problem and how to construct a state-space realization of the generalized plant. For convenience, Glover and Doyle's suboptimal H^∞ controller formulas [8] will be briefly reviewed. These suboptimal H^∞ controller formulas can be easily transformed into a descriptor form such that an optimal H^∞ controller can be constructed without numerical difficulties if the optimal H^∞ norm is known. The optimal H^∞ controllers, with very few exceptions, have direct feed through terms and therefore infinite bandwidth. Hence, control

engineers may prefer strictly proper suboptimal H^∞ controllers to the optimal ones. However, knowing the optimal H^∞ norm is important in determining which suboptimal controller to be chosen in practical design.

Unlike the well-known LQG Riccati equations [11], the H^∞ Riccati equations are parameter dependent and may not have solutions for some values of the parameter γ. The H^∞ optimization design can only be achieved by iteratively searching for the optimal H^∞ norm, which is the smallest γ such that an H^∞ controller exists. This search is the major computational burden in H^∞ design, and bisection method has been widely used in the design procedure. As it is known, the bisection method converges slowly and therefore is inefficient. To improve the convergence rate, we need to develop a more efficient search scheme which may require a better understanding of the H^∞ Riccati equations.

To investigate the properties of the two H^∞ Riccati equations, first of all in Chapter 3 we will consider a special case with which the D_{11} matrix of the generalized plant is assumed zero. With this assumption, the two H^∞ Riccati equations are greatly simplified. We will investigate the structure of the γ-domain where the H^∞ Riccati solutions exist and/or positive semidefinite, and reveal some useful properties of the Riccati solutions on the γ-domain, such as *continuity*, *monotonicity*, and *convexity* with respect to γ. By these properties, quadratically convergent algorithms can be easily developed to compute the optimal H^∞ norm of the closed-loop system.

The assumption that the D_{11} matrix of the generalized plant is zero can hardly be satisfied by many practical problems. In the general case with D_{11} nonzero, the H^∞ Riccati Hamiltonains are much more complicated functions of γ than those in the special case and hence make the investigation of the properties extremely difficult. In Chapter 4, several decompositions and some key intermediate variables are employed to tackle the complexities and to show that the monotonicity and convexity properties also hold for the general case. Based on these properties, a quadratically convergent algorithm can be easily developed to compute the optimal H^∞ norm. The algorithm is presented in Chapter 5.

In Chapter 6, a fighter aircraft longitudinal flight control design problem is used to demonstrate an application of μ-synthesis technique. With the robust controller obtained, simulations and analyses are performed on the closed loop system. For the example we considered, the order of the μ-synthesis controller can be reduced to that of the plant without degrading the performance.

From Chapters 7 to 9, three controller reduction approaches are presented. A controller reduction approach based on the closed-loop controllability and observability gramians is proposed in Chapter 7. The motivation is to retain the input/output relationship of the closed-loop system while the order of the controller is reduced. A structured transformation on the gramians is employed to achieve this objective.

In Chapter 8, a parametrization method is proposed for reduced-order H^∞ controller design. By using the state-space H^∞ optimization approach, one can characterize the set of all (sub)optimal controllers in terms of a parameter matrix. It is shown that there always exists a suitable parameter matrix such that a reduced-order H^∞ controller can be constructed. The procedure for obtaining such a parameter matrix is also presented in this chapter.

In Chapter 9, the reduced compensator design issue was tackled from an observer-based compensator point of view. It develops two properties related to observer based controller parametrization and pole placement. It shows that the poles of the closed-loop system with the observer-based controller parametrization are the regulator poles, the observer poles, together with the poles of the added stable parameter matrix. If the controller is realized by a *minimal realization*, the closed-loop poles will include all the poles of the added stable parameter matrix and a subset of the regulator and the observer poles. It also points out that there exist parameter matrices which would render the controller non-minimal, thereby yielding reduced order controllers. Such order-reducing parameter matrices - both static and dynamic - have been characterized herein. With such a parametrization available, one could then choose a K(s) in order to best approximate large order controllers, such as the H^∞ compensators, by a lower order controller.

Incorporating such a parametrization in a design procedure requires the identification of poles that are to be rendered uncontrollable and unobservable. Part of the research has focused on this issue. In particular, we have developed a means of identifying dominant modes of mechanical systems based on an L_2 norm which is summarized in Chapter 10.

Chapter 11 is the conclusion. The work for future research will also be briefly described in Chapter 11.

CHAPTER 2

FORMULATION OF H^∞ CONTROL PROBLEMS

2.1 The H^∞ Optimization Problem

Most control problems can be formulated as the following H^∞ optimization problem. A detailed procedure of the formulation will be discussed in Section 2.3. In the H^∞ optimization problem formulation, the system under consideration is described as follows,

$$\begin{bmatrix} z(s) \\ y(s) \end{bmatrix} = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} \begin{bmatrix} v(s) \\ u(s) \end{bmatrix} := G(s) \begin{bmatrix} v(s) \\ u(s) \end{bmatrix} \tag{2-1}$$

where $G_{11}(s) \in \mathbf{R}(s)^{p1xm1}$, $G_{12}(s) \in \mathbf{R}(s)^{p1xm2}$, $G_{21}(s) \in \mathbf{R}(s)^{p2xm1}$, and $G_{22}(s) \in \mathbf{R}(s)^{p2xm2}$. $\mathbf{R}(s)^{pxq}$ is the set of pxq proper rational matrices with real coefficients. In (2-1), z, y, v, and u are the controlled output, the measured output, the exogenous input, and the control input respectively. The controlled output vector z usually includes the error signal and a weighted control input. The exogenous input v contains the disturbances, the noises, and the commands. The measured output vector y consists of all the signals which can be measured and available for feedback. Through the control input u, the behavior of the system can be modified. The vector y will be used as the input to a controller K(s) and the output of K(s) will be connected to the control input u. That is,

$$u(s) = K(s)\, y(s)\,. \tag{2-2}$$

The standard H^∞ optimization problem is the problem of finding a proper controller K(s) such that the closed-loop system is internally stable and $\| \mathcal{F}_\ell(G,K) \|_\infty$ is minimized where

$$\mathcal{F}_\ell(G,K)\,(s) = G_{11}(s) + G_{12}(s)\,K(s)\,[\,I - G_{22}(s)\,K(s)\,]^{-1}\,G_{21}(s)\,. \tag{2-3}$$

That is, $\mathcal{F}_\ell(G,K)\,(s)$, the lower linear fractional transformation, is the transfer function of the closed-loop system from v to z.

Let

$$G(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] \tag{2-4}$$

be a realization of the generalized plant G(s) and $A \in \mathbb{R}^{n\times n}$.

In [8], Glover and Doyle assume that the realization of the generalized plant G(s) is given by (2-4) with the following assumptions.

(i) (A, B_2) is stabilizable and (C_2, A) is detectable.

(ii) rank $D_{12} = m_2$, rank $D_{21} = p_2$.

(iii) $D_{12} = \begin{bmatrix} 0 \\ I \end{bmatrix}$, $D_{21} = [0 \quad I]$, and D_{11} is partitioned as

$\begin{bmatrix} D_{1111} & D_{1112} \\ D_{1121} & D_{1122} \end{bmatrix}$ with $D_{1122} \in \mathbb{R}^{p2\times m2}$,

(iv) $D_{22} = 0$ (this can be removed, for details please refer to [8]).

(v) $\text{rank} \begin{bmatrix} j\omega I - A & B_2 \\ C_1 & D_{12} \end{bmatrix} = n + m_2 \quad \forall\, \omega \in \mathbb{R}$.

(vi) $\text{rank} \begin{bmatrix} j\omega I - A & B_1 \\ C_2 & D_{21} \end{bmatrix} = n + p_2 \quad \forall\, \omega \in \mathbb{R}$.

Define two Hamiltonian matrices as follows,

$$H_\infty(\gamma) := \begin{bmatrix} A & 0 \\ -C_1^T C_1 & -A^T \end{bmatrix} - \begin{bmatrix} B \\ -C_1^T D_{1\bullet} \end{bmatrix} R^{-1} \begin{bmatrix} D_{1\bullet}^T C_1 & B^T \end{bmatrix} \tag{2-5a}$$

and

$$J_\infty(\gamma) := \begin{bmatrix} A^T & 0 \\ -B_1 B_1^T & -A \end{bmatrix} - \begin{bmatrix} C^T \\ -B_1 D_{\bullet 1}^T \end{bmatrix} \tilde{R}^{-1} \begin{bmatrix} D_{\bullet 1} B_1^T & C \end{bmatrix} \tag{2-5b}$$

where

$$D_{1\bullet} = \begin{bmatrix} D_{11} & D_{12} \end{bmatrix}, \qquad D_{\bullet 1} = \begin{bmatrix} D_{11} \\ D_{21} \end{bmatrix}, \tag{2-5c}$$

and

$$R = D_{1\bullet}^T D_{1\bullet} - \begin{bmatrix} \gamma^2 I_{m1} & 0 \\ 0 & 0 \end{bmatrix}, \quad \tilde{R} := D_{\bullet 1} D_{\bullet 1}^T - \begin{bmatrix} \gamma^2 I_{p1} & 0 \\ 0 & 0 \end{bmatrix}. \tag{2-5d}$$

Then the following theorem shows an easy way to construct a suboptimal stabilizing controller such that $\|\mathfrak{F}_{\ell}(G,K)\|_{\infty} < \gamma$ where $\mathfrak{F}_{\ell}(G,K)$ is the closed-loop transfer matrix from v to z.

Theorem 2.1: [8]

There exists a stabilizing controller such that $\|\mathfrak{F}_{\ell}(G,K)\|_{\infty} < \gamma$ if and only if the following three conditions hold.

(i) $\gamma > \max\left(\bar{\sigma}\begin{bmatrix} D_{1111} & D_{1112} \end{bmatrix}, \bar{\sigma}\begin{bmatrix} D_{1111}^T & D_{1121}^T \end{bmatrix} \right)$ (2-6a)

(ii) $H_{\infty}(\gamma) \in \mathrm{dom(Ric)}$ and $X_{\infty}(\gamma) := \mathrm{Ric}[H_{\infty}(\gamma)] \geq 0$. (2-6b)

(iii) $J_{\infty}(\gamma) \in \mathrm{dom(\ Ric\ )}$ with $Y_{\infty}(\gamma) = \mathrm{Ric}[J_{\infty}(\gamma)] \geq 0$. (2.6c)

(iv) $\rho[X_{\infty}(\gamma)Y_{\infty}(\gamma)] < \gamma^2$. (2-6d)

Moreover, when these conditions hold, one such controller is

$$K_{sub}(s) = \left[\begin{array}{c|c} A_k & B_k \\ \hline C_k & D_k \end{array}\right] \tag{2-7a}$$

where

$$D_k = -D_{1121}D_{1111}^T(\gamma^2 I - D_{1111}D_{1111}^T)^{-1}D_{1112} - D_{1122} \tag{2-7b}$$

$$C_k = \{F_2 - D_k(C_2+F_{12})\}Z \tag{2-7c}$$

$$B_k = -H_2 + (B_2+H_{12})D_k \tag{2-7d}$$

$$A_k = A+HC+(B_2+H_{12})C_k \tag{2-7e}$$

$$Z := (I - \gamma^{-2}YX)^{-1} \tag{2-7f}$$

$$F^T = \begin{bmatrix} F_{11}^T & F_{12}^T & F_2^T \end{bmatrix} = -(XB + C_1^T D_{1\bullet})R^{-1} \tag{2-7g}$$

$$H = \begin{bmatrix} H_{11} & H_{12} & H_2 \end{bmatrix} = -(YC^T + B_1 D_{\bullet 1}^T)\tilde{R}^{-1} \tag{2-7h}$$

and $F_{11} \in \mathbb{R}^{(m_1-p_2)xn}$, $F_{12} \in \mathbb{R}^{p_2xn}$, $F_2 \in \mathbb{R}^{m_2xn}$, $H_{11} \in \mathbb{R}^{nx(p_1-m_2)}$, $H_{12} \in \mathbb{R}^{nxm_2}$, $H_2 \in \mathbb{R}^{nxp_2}$.

In the above theorem, condition (ii) means that there exist positive semi-definite solutions X and Y to the algebraic Riccati equations corresponding to the Hamiltonians $H_\infty(\gamma)$ and $J_\infty(\gamma)$ respectively. Condition (iii) means that the spectral radius of XY is less than γ^2.

The above theorem provides an easy way to construct a stabilizing suboptimal controller such that $\|\mathcal{F}_\ell(G,K)\|_\infty < \gamma$. The order of the suboptimal controller can be the same as that of the plant G(s). The major computation involved is the solution of two H^∞ Riccati equations which are easy to solve if solutions exist.

Theorem 2.1 can also be used to compute the optimal H^∞ norm and to construct an optimal H^∞ controller. Algorithms for computing the optimal H^∞ norm will be discussed in Chapter 4 and the construction of an optimal H^∞ controller will be given in Section 2.2. Section 2.3 will explain how to formulate a robust control problem as a standard H^∞ optimization problem.

2.2 Construction of Optimal H^∞ Controllers

The optimum can occur in three cases. In case (1), the optimum occurs at the smallest γ such that the two H^∞ Riccati equations have stabilizing solutions X and Y, i.e., these X and Y happen to be positive semi-definite and $\rho(XY) < \gamma^2$. Case (2) occurs when Y=0 (or X=0) for all γ and the optimal H^∞ norm is the smallest γ such that X (or Y) is positive semi-definite. The most likely one to happen most of the time is case (3) in which the optimal H^∞ norm is the γ such that the two H^∞ Riccati equations have positive semi-definite stabilizing solutions X and Y and $\rho(XY) = \gamma^2$ where $\rho(XY)$ is the spectral radius of XY.

From Theorem 2.1, a suboptimal H^∞ controller can be easily constructed. However, as γ approaches to the optimum we will encounter the inversion of a singular matrix except case (1) which seldom occurs. To eliminate the numerical difficulty, Safonov et. al. [12] rederived the optimal controller formulas in a descriptor form (or generalized state-space representation).

The formulas in (2-7a) - (2-7h) can also be written in a descriptor form after slight rearrangement. We will firstly consider case (3) which occurs much more frequently than the other two cases. When γ reaches the optimum, γ_o, which satisfies $\gamma_o^2 = \rho[X(\gamma_o)Y(\gamma_o)]$,

the matrix Z in (2-7f) will become infinity since the matrix $I-\gamma_o^2 Y(\gamma_o)X(\gamma_o)$ is singular. If we try to apply the formulas (2-7a) - (2-7h) directly to construct an optimal H^∞ controller, a numerical difficulty will arise in the implementation of the A_k and C_k matrices. We will rearrange these formulas such that an optimal H^∞ controller can be constructed without any numerical difficulty.

The dual system of the realization in (2-7a) can be easily rewritten in a descriptor form. The state equation (generalized state equation) of the descriptor representation can be split into two set of equations: one involves first derivative of some state variables and the other is just an algebraic equation. The state variables which have no derivative in the equations can be eliminated and then we have a lower order state space representation for the dual system. The dual of the dual system is identical to the original and therefore we have an optimal H^∞ controller as follows:

$$K_{opt}(s) = \left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & D_c \end{array}\right] \tag{2-8}$$

where

$$A_c = [\, V_1^T A_D U_1 - V_1^T A_D U_2 (V_2^T A_D U_2)^\dagger V_2^T A_D U_1 \,] \Sigma_1^{-1} \tag{2-9a}$$

$$B_c = V_1^T B_D - V_1^T A_D U_2 (V_2^T A_D U_2)^\dagger V_2^T B_D \tag{2-9b}$$

$$C_c = [\, C_D U_1 - C_D U_2 (V_2^T A_D U_2)^\dagger V_2^T A_D U_1 \,] \Sigma_1^{-1} \tag{2-9c}$$

$$D_c = D_k - C_D U_2 (V_2^T A_D U_2)^\dagger V_2^T B_D \tag{2-9d}$$

and

$$B_D = -H_2 + (B_2 + H_{12}) D_k \tag{2-10a}$$

$$C_D = F_2 - D_k (C_2 + F_{12}) \tag{2-10b}$$

$$A_D = (B_2 + H_{12}) C_D + (A + HC) E_D \tag{2-10c}$$

$$E_D = I - \gamma_o^{-2} X(\gamma_o) Y(\gamma_o) \tag{2-10d}$$

Σ_1, U_1, U_2, V_1, and V_2 are obtained from the singular value decomposition of E_D, i.e.,

$$E_D = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix} \tag{2-11}$$

Similar procedure can be used to construct optimal H^∞ controllers for Case (2) and

the formulas in Theorem 2.1 can be used to construct optimal H^∞ controllers for Case (1).

2.3 Formulation of H^∞ Optimization Problems

Many control problems can be formulated as the standard H^∞ optimization problem. For the purpose of demonstration, two examples are given in the following. The first is a mixed-sensitivity optimization problem; the second is a disturbance reduction problem with measurement noise.

A. Mixed-Sensitivity Optimization Problem

Consider the following system:

$$y(s) = P(s)u(s) + v(s) \tag{2-12a}$$

$$u(s) = K(s)y(s) \tag{2-12b}$$

where v(s) is disturbance, y(s) is output and K(s) is controller to be designed. It is well known that a smaller $\|(I-PK)^{-1}\|_\infty$ means a better disturbance attenuation, whereas a smaller $\|PK(I-PK)^{-1}\|_\infty$ implies a better robust stability. Unfortunately, the H^∞ norms of $(I-PK)^{-1}$ and $PK(I-PK)^{-1}$ may not be made small at the same time. If we make one of them smaller then the other will become larger. To have a trade-off between these two quantities, Kwakernaak [3] formulated the mixed-sensitivity problem as the problem of finding a controller K(s) which stabilizes the closed-loop system and minimizes $\|\Phi\|_\infty$ where Φ is given by

$$\Phi = \begin{bmatrix} W_1(I-PK)^{-1} \\ W_2PK(I-PK)^{-1} \end{bmatrix} \tag{2-13}$$

W_1 and W_2 are weighting matrices chosen by the designer according to the concrete situation. In other words, they depend on the characters of the disturbances and system uncertainties. Usually, the disturbances occur most likely at low frequency, therefore $W_1(s)$ is chosen to be a low-pass filter to emphasize the error energy at low frequency. The plant uncertainty is also frequency-dependent; the higher the frequency is, the larger the uncertainties become. Hence, $W_2(s)$ is usually chosen to be a improper transfer function (but $W_2P(s)$ has to be a proper transfer function), which is analytic in closed right half plane. In the following, we assume that $W_1(s)$ is strictly proper, $W_2(s)$ is a polynomial such that $W_2P(s)$ remains proper and both of them are analytic in closed right half plane.

The problem of finding a K(s) which stabilizes the closed-loop system and minimizes $\|\Phi\|_\infty$ can be rearranged into the standard H^∞ optimization problem. Consider the following system:

$$\begin{bmatrix} z_1 \\ z_2 \\ y \end{bmatrix} = \left[\begin{array}{c|c} W_1 & W_1 P \\ 0 & W_2 P \\ \hline I & P \end{array}\right] \begin{bmatrix} v \\ u \end{bmatrix} \tag{2-14a}$$

$$u = K y \tag{2-14b}$$

It is easy to show that the matrix Φ defined by (2-13) is just the transfer function from v to $[z_1^T \; z_2^T]^T$ of the closed-loop system (2-14). Comparing (2-14a) with (2-1), we can see that

$$G_{11} = \begin{bmatrix} W_1 \\ 0 \end{bmatrix}, \qquad G_{12} = \begin{bmatrix} W_1 P \\ W_2 P \end{bmatrix},$$
$$G_{21} = I, \qquad G_{22} = P. \tag{2-15}$$

If P, W_2P, and W_1 have state-space realizations as follows

$$P = \left[\begin{array}{c|c} A_p & B_p \\ \hline C_p & D_p \end{array}\right], \quad W_2 P = \left[\begin{array}{c|c} A_p & B_p \\ \hline C_{w2} & D_{w2} \end{array}\right], \quad W_1 = \left[\begin{array}{c|c} A_{w1} & B_{w1} \\ \hline C_{w1} & D_{w1} \end{array}\right] \tag{2-16}$$

Then the generalized plant G(s) has a state space realization as shown in (2-4) with

$$A = \begin{bmatrix} A_p & 0 \\ B_{w1} C_p & A_{w1} \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ B_{w1} \end{bmatrix}, \quad B_2 = \begin{bmatrix} B_p \\ B_{w1} D_p \end{bmatrix}$$

$$C_1 = \begin{bmatrix} D_{w1} C_p & C_{w1} \\ C_{w2} & 0 \end{bmatrix}, \quad D_{11} = \begin{bmatrix} D_{w1} \\ 0 \end{bmatrix}, \quad D_{12} = \begin{bmatrix} D_{w1} D_p \\ D_{w2} \end{bmatrix}$$

$$C_2 = \begin{bmatrix} C_p & 0 \end{bmatrix}, \qquad D_{21} = I, \qquad D_{22} = D_p \tag{2-17}$$

Note that because W_2 is a polynomial, the A-matrix of W_2P is same as that of P.

B. Disturbance Reduction Problem

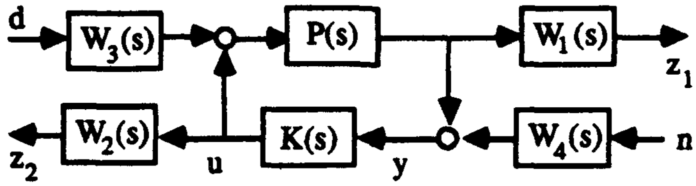


Fig.2.1 A disturbance attenuation problem

Consider the feedback system shown in Fig.2.1. P(s) is a given plant, $W_i(s)$, $i=1,2,3,4$ are weighting matrices, and K(s) is the controller to be designed. The disturbance and noise are the outputs of W_3 and W_4 driven by d and n respectively. z_1 ⁚ the weighted error response; z_2 is the weighted control input. Let $z^T = [z_1^T \; z_2^T]^T$, $v^T = [d$ $n^T]^T$ and assume that v is unknown but with its energy bounded by unity. The objective is to find a controller K(s) which stabilizes the closed-loop system and minimizes the worst $\|z\|_2$, i.e., minimizes the H^∞ norm of T_{zv}, the closed-loop transfer function from v to z. T_{zv} is given by

$$T_{zv} = \begin{bmatrix} W_1P(I-KP)^{-1}W_3 & W_1PK(I-PK)^{-1}W_4 \\ W_2KP(I-KP)^{-1}W_3 & W_2K(I-PK)^{-1}W_4 \end{bmatrix} \tag{2-18}$$

Note that $W_1PK(I-PK)^{-1}W_4$ and $W_2KP(I-KP)^{-1}W_3$) are the output and input complementary sensitivity functions. Their H^∞ norms indicate the stability robustness of the closed-loop system for the multiplicative plant uncertainty introduced at the output and input respectively. $W_2K(I-PK)^{-1}W_4$ is the control complementary sensitivity function whose H^∞ norm indicates the stability robustness of the closed-loop system for additive plant uncertainty. Hence, reducing $\|T_{zv}\|_\infty$ will also improve the robust stability of the closed-loop system.

It is easy to verify that the generalized plant of the system can be expressed as:

$$\begin{bmatrix} z_1 \\ z_2 \\ \hline y \end{bmatrix} = \left[\begin{array}{cc|c} W_1PW_3 & 0 & W_1P \\ 0 & 0 & W_2 \\ \hline PW_3 & W_4 & P \end{array}\right] \begin{bmatrix} d \\ n \\ \hline u \end{bmatrix} \tag{2-19}$$

That is,

$$G_{11} = \begin{bmatrix} W_1PW_3 & 0 \\ 0 & 0 \end{bmatrix}, \quad G_{12} = \begin{bmatrix} W_1P \\ W_2 \end{bmatrix}$$
$$G_{21} = \begin{bmatrix} PW_3 & W_4 \end{bmatrix}, \quad G_{22} = P. \tag{2-20}$$

If P, W_i, i=1,2,3,4 have state-space realizations as follows:

$$P = \left[\begin{array}{c|c} A_p & B_p \\ \hline C_p & D_p \end{array}\right], \quad W_i = \left[\begin{array}{c|c} A_{wi} & B_{wi} \\ \hline C_{wi} & D_{wi} \end{array}\right], \quad i=1,2,3,4 \tag{2-21}$$

Then the generalized plant G(s) has a state space realization as shown in (2-4) with

$$A = \begin{bmatrix} A_p & 0 & 0 & B_pC_{w3} & 0 \\ B_{w1}C_p & A_{w1} & 0 & B_{w1}D_pC_{w3} & 0 \\ 0 & 0 & A_{w2} & 0 & 0 \\ 0 & 0 & 0 & A_{w3} & 0 \\ 0 & 0 & 0 & 0 & A_{w4} \end{bmatrix}, \quad B_1 = \begin{bmatrix} B_pD_{w3} & 0 \\ B_{w1}D_pD_{w3} & 0 \\ 0 & 0 \\ B_{w3} & 0 \\ 0 & B_{w4} \end{bmatrix}, \quad B_2 = \begin{bmatrix} B_p \\ B_{w1}D_p \\ B_{w2} \\ 0 \\ 0 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} D_{w1}C_p & C_{w1} & 0 & D_{w1}D_pC_{w3} & 0 \\ 0 & 0 & C_{w2} & 0 & 0 \end{bmatrix}, \quad D_{11} = \begin{bmatrix} D_{w1}D_pD_{w3} & 0 \\ 0 & 0 \end{bmatrix}, \quad D_{12} = \begin{bmatrix} D_{w1}D_p \\ D_{w2} \end{bmatrix}$$

$$C_2 = \begin{bmatrix} C_p & 0 & 0 & D_pC_{w3} & C_{w4} \end{bmatrix}, \quad D_{21} = \begin{bmatrix} D_pD_{w3} & D_{w4} \end{bmatrix}, \quad D_{22} = D_p. \tag{2-22}$$

Above {A,B,C,D} is the state-space representation for the generalized plant G(s).

CHAPTER 3

STRUCTURE OF THE γ-DOMAIN FOR THE DGKF H^∞ RICCATI SOLUTIONS

3.1 Introduction

The major computational burden involved in the two γ-dependent H^∞ Riccati equation approach is the computation of the optimal H^∞ norm. To develop efficient algorithm for computing the optimal H^∞ norm, it is necessary to investigate how the Riccati solutions vary as a function of the paramter γ.

There have been several attempts to address these issues. Pandey *et al.*'s hybrid gradient-bisection method [13] and Chang *et al.*'s double secant and bisection method [14] were proposed for the computation of the optimal H^∞ norm. In these two papers, a conjecture was made that the spectral radius of the product of the two Riccati solutions is a convex function of γ^2. Base on this conjecture, improved search schemes were developed. However, since there was no proof for this conjecture, bisection was used in these two algorithms as a supplement to guarantee convergence. In [15], Scherer considered the inverse of the H^∞ Riccati solutions, defined a new independent variable $\mu = \gamma^{-2}$, and showed that these inverses are concave functions of μ in matrix sense. Based on this fact, a Newton-like algorithm was proposed to compute the optimal H^∞ norm when an appropriate starting point is available. The convexity of the spectral radius of the product of the two Riccati solutions with respect to γ was first shown by Li and Chang [16]. With this convexity property, the optimal H^∞ norm can be easily computed by a quadratically convergent algorithm if an appropriate starting point is available. However, it is not trivial to obtain such a starting point that can guarantee the convergence. Furthermore, no general and rigorous analysis is available for the γ-dependent H^∞ Riccati equations.

In this chapter, we investigate the structure of the γ-domain where the H^∞ Riccati solutions exist and/or positive semidefinite, and reveal some useful properties of the H^∞ Riccati solutions on the γ-domain, such as continuity, monotonicity, and convexity with

respect to γ. These results provide a better understanding of the state space approach to H^∞ optimization problems. Beside their theoretical value, these properties can be employed to develop efficient algorithms for partitioning the γ-domain, finding the smallest γ such that the two Riccati solutions are positive semidefinite, and computing the optimal H^∞ norm. The chapter is organized as follows. Section 3.2 lists the notations and reviews preliminaries. In Section 3.3, we investigate the structure of γ-domain of the H^∞ Riccati solutions. The continuity, monotonicity, and convexity properties of the Riccati solutions are discussed in Section 3.4.

3.2 Notations and Preliminaries

$\mathbb{R}$	The set of real numbers.
$\mathbb{R}_+$	The set of positive real numbers.
$\inf(S)$	The infimum of the set S.
$\sup(S)$	The supremum of the set S.
$\mathbb{C}^{m\times n}$	The set of mxn matrices whose entries are complex numbers.
$\mathbb{R}^{m\times n}(s)$	The set of mxn transfer matrices.
I	Identity matrix.
I_n	n-dimensional identity matrix.
A^T or A'	Transpose of a matrix A.
A^*	Conjugate transpose of a matrix A.
$G^\sim(s)$	$G^T(-s)$.
$A\geq 0$	A is a positive semidefinite matrix.
$A>0$	A is a positive definite matrix.
$\lambda(A)$	An arbitrary eigenvalue of A.
$\lambda_i(A)$	The ith eigenvalue of A.
$\lambda_{max}(A)$	The maximal eigenvalue of A.
$\lambda_{min}(A)$	The minimal eigenvalue of A.
$\rho(A)$	The spectral radius of A.

σ_{max} or $\bar{\sigma}$	The maximal singular value of A.
$A(\gamma)$	The matrix A is considered as a function of $\gamma \in \mathbb{R}_+$.
$\dot{A}(\gamma)$	The first derivative of $A(\gamma)$ with respect to γ, i.e., $\frac{dA(\gamma)}{d\gamma}$.
$\ddot{A}(\gamma)$	The second derivative of $A(\gamma)$ with respect to γ, i.e., $\frac{d^2A(\gamma)}{d\gamma^2}$.
$\dot{A}^{-1}(\gamma)$	The first derivative of $A^{-1}(\gamma)$, i.e., $\frac{d}{d\gamma}(A^{-1}(\gamma))$.
$\ddot{A}^{-1}(\gamma)$	The second derivative of $A^{-1}(\gamma)$ with respect to γ, i.e., $\frac{d^2}{d\gamma^2}(A^{-1}(\gamma))$.

Throughout the chapter, the notation $P(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ or $P(s) = \{A, B, C, D\}$ is used to represent a state space realization of a system whose transfer function is $P(s) = C(sI - A)^{-1}B + D$. The H^∞ norm of P(s) is defined as

$$\| P \|_\infty := \sup_\omega \bar{\sigma}\,[P(j\omega)]$$

Many control problems can be formulated as the following standard H^∞ optimization problem. In the standard H^∞ optimization problem formulation, the system representation is rearranged as follows.

$$\begin{bmatrix} z(s) \\ y(s) \end{bmatrix} = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} \begin{bmatrix} w(s) \\ u(s) \end{bmatrix} := G(s) \begin{bmatrix} w(s) \\ u(s) \end{bmatrix} \tag{3-1}$$

where z, y, w, and u are the controlled output, the measured output, the exogenous input, and the control input respectively. The controlled output vector z usually includes the error signal and a weighted control input. The exogenous input w contains the disturbances, the noises, and the commands. The measured output vector y consists of all the signals which can be measured and available for feedback. Through the control input u, the behavior of the system can be modified. The vector y will be used as the input to a controller K(s) and the output of K(s) will be connected to the control input u. That is,

$$u(s) = K(s)\, y(s) \tag{3-2}$$

The standard H^∞ optimization problem is the problem of finding a proper controller K(s) such that the closed-loop system is internally stable and $\| T_{zw} \|_\infty$ is minimized where

$$T_{zw}(s) = G_{11}(s) + G_{12}(s)\, K(s)\, [\, I - G_{22}(s)\, K(s)\,]^{-1}\, G_{21}(s) \tag{3-3}$$

That is, $T_{zw}(s)$ is the transfer function of the closed-loop system from w to z.

Let a realization of the generalized plant G(s) be

$$G(s) = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right] \tag{3-4}$$

with the following assumptions:

(i) Both $G_{12}(s)$ and $G_{21}(s)$ have no transmission zeros on the $j\omega$ axis. (3-5a)

(ii) (A, B_2) is stabilizable and (C_2, A) is detectable. (3-5b)

(iii) $D_{12}'\,[\, C_1 \;\; D_{12}\,] = [\, 0 \;\; I\,]$ (3-5c)

(iv) $\begin{bmatrix} B_1 \\ D_{21} \end{bmatrix} D_{21}' = \begin{bmatrix} 0 \\ I \end{bmatrix}$ (3-5d)

The two Riccati equations involved in the state space approach are:

$$A' X_\infty + X_\infty A + X_\infty (\gamma^{-2} B_1 B_1' - B_2 B_2')\, X_\infty + C_1' C_1 = 0 \tag{3-6a}$$

and

$$A\, Y_\infty + Y_\infty A' + Y_\infty (\gamma^{-2} C_1' C_1 - C_2' C_2)\, Y_\infty + B_1 B_1' = 0. \tag{3-6b}$$

The following theorem by Doyle et. al. characterizes all suboptimal stabilizing controllers such that $\| T_{zw} \|_\infty < \gamma$.

Theorem 3.1: [7] There exists a stabilizing controller such that $\|T_{zw}\|_\infty < \gamma$ if and only if the following three conditions hold:

(i) There exists a positive semidefinite stabilizing solution $X_\infty(\gamma)$ to (3-6a).

(ii) There exists a positive semidefinite stabilizing solution $Y_\infty(\gamma)$ to (3-6b).

(iii) $\rho[X_\infty(\gamma) Y_\infty(\gamma)] < \gamma^2$.

When these conditions hold, one such controller is

$$K_{sub}(s) = \left[\begin{array}{c|c} A_k & -ZL \\ \hline F & 0 \end{array}\right]$$

where

$$A_k = A + \gamma^{-2}B_1B_1'X_\infty + B_2F + ZLC_2$$
$$F = -B_2'X_\infty,\ L = -Y_\infty C_2',\ E = (I - \gamma^{-2}Y_\infty X_\infty),\ Z = (I - \gamma^{-2}Y_\infty X_\infty)^{-1}.$$

The major computational task of H^∞ design is to solve the two Riccati equations, eq.(3-6a&b). From [7], the stabilizing solution X_∞ and Y_∞ can be solved via their corresponding Hamiltonian matrices

$$H_\infty(\gamma) = \begin{bmatrix} A & \gamma^{-2}B_1B_1' - B_2B_2' \\ -C_1'C_1 & -A' \end{bmatrix} \tag{3-7a}$$

$$J_\infty(\gamma) = \begin{bmatrix} A' & \gamma^{-2}C_1'C_1 - C_2'C_2 \\ -B_1B_1' & -A \end{bmatrix}. \tag{3-7b}$$

Namely, if we let

$$T_\infty = \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} \tag{3-8}$$

be a modal matrix for H_∞ corresponding to all the eigenvalues on the open left half of s-plane, then

$$X_\infty = T_2T_1^{-1}, \tag{3-9}$$

is the unique stabilizing solution to eq.(3-6a) such that $(\gamma^{-2}B_1B_1' - B_2B_2')X_\infty$ is stable [1,7]. It is well-known [7] that the stabilizing X_∞ exists if and only if (1) H_∞ has no $j\omega$-axis eigenvalues, and (2) T_1 is invertible. These two conditions are called stability condition and complementary condition respectively in [7]. We have the same arguments for Y_∞, the stabilizing solution to eq.(3-6b).

3.3 Structure of the γ-Domain of H^∞ Riccati Solutions

It is well-known that for the standard LQG Riccati equations [17], positive semidefinite stabilizing solutions exist if the system is stabilizable and detectable. Unfortunately, the H^∞ Riccati equations are more complicated because they are parameter dependent and their quadratic-term coefficient matrices may not be sign definite.

For a sufficient large γ, a suboptimal H^∞ controller always exists. This is due to the fact that as $\gamma \to \infty$, the H^∞ Riccati equations eq.(3-6a&b) become the standard LQG Riccati equations (the quadratic terms become sign-definite), therefore both X_∞ and Y_∞ exist and

are positive semidefinite. Meanwhile, since γ is sufficiently large, the third condition of Theorem 3.1 is also satisfied. However, as γ decreases, the three conditions of Theorem 2.1 may not hold any more. For instance, X_∞ may not exist or fail to be positive semidefinite for some small γ. Thus the following problems arise:

1. How small the γ could be such that the Riccati equations have stabilizing solutions?
2. What is the structure of γ-domain where the solutions exist?
3. How to compute the optimal γ, which is defined as

$$\gamma_{opt} = \inf\{\gamma\text{: all the three conditions of Theorem 3.1 hold}\}.$$

We will answer these questions by starting the investigation of the structure of γ-domain for $X_\infty(\gamma)$.

Theorem 3.2: If we define

$$\alpha_x := \inf\{\gamma : \gamma \in \mathbb{R}_+ \text{ and } X_\infty(\gamma) \text{ exists}\}$$
$$\beta_x := \inf\{\gamma : \gamma \in \mathbb{R}_+ \text{ and } X_\infty(\gamma) \text{ is positive semidefinite}\},$$

then we have the following results:

a) On $(\alpha_x, +\infty)$, $X_\infty(\gamma)$ is well defined almost everywhere;

b) On $(\beta_x, +\infty)$, $X_\infty(\gamma)$ is well defined and positive semidefinite everywhere;

c) $\alpha_x \le \beta_x$.

Fig.3.1 gives an illustration for the theorem. One can see that the γ-domain for $X_\infty(\gamma)$ consists of three parts which are bounded by α_x and β_x. For any γ in Region c, $X_\infty(\gamma)$ is positive semidefinite; whereas in region a, $X_\infty(\gamma)$ does not exist at all. In Region b, although $X_\infty(\gamma)$ is not positive semidefinite, it exists everywhere except for some possible isolated points.

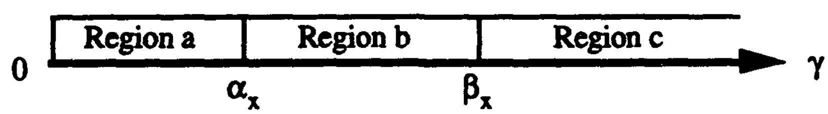


Fig. 3.1 γ-domain of $X_\infty(\gamma)$.

In order to simplify the proof of Theorem 3.2, we will, without loss of generality, strengthen the assumption on (C_1, A) from detectable to observable. To show that this can be done, we assume that (C_1, A) is not observable. Then one can always restrict the problem into the observable subspace by finding an orthogonal matrix [18]

$$U = [\, U_1 \quad U_2 \,] \tag{3-10}$$

such that

$$U'AU = \begin{bmatrix} U_1'AU_1 & 0 \\ U_2'AU_1 & U_2'AU_2 \end{bmatrix}, \tag{3-11}$$

$$U'B = \begin{bmatrix} U_1'B_1 & U_1'B_2 \\ U_2'B_1 & U_2'B_2 \end{bmatrix}, \tag{3-12}$$

and

$$C_1U = [C_1U_1 \quad 0] \tag{3-13}$$

with $(C_1U_1, U_1'AU_1)$ observable. It is easy to verify that the solution to eq.(3-6a) can be expressed as

$$X_\infty(\gamma) = U \begin{bmatrix} X(\gamma) & 0 \\ 0 & 0 \end{bmatrix} U', \tag{3-14}$$

where $X(\gamma)$ is the stabilizing solution to the eq.(3-6a) with (A, B_1, B_2, C_1) replaced by $(U_1'AU_1, U_1'B_1, U_1'B_2, C_1U_1)$. Note that, since the matrix U in eq.(3-14) is independent of γ, $X_\infty(\gamma)$ exists if and only if $X(\gamma)$ does, and $X_\infty(\gamma)$ is positive semidefinite if and only if $X(\gamma)$ is. Hence, in the following, we will assume that (C_1, A) is observable and use $X_\infty(\gamma)$ and $X(\gamma)$ interchangeably without loss of generality.

To prove Theorem 3.2, we need the following lemmas.

Lemma 3.1 [7]: The stabilizing solution to the H^∞ Riccati equation eq.(3-6a) can be written in terms of two LQG Riccati solutions:

$$X_\infty(\gamma) = X_2[\, X_2 - W(\gamma)\,]^{-1}X_2, \tag{3-15}$$

where X_2 is the stabilizing solution to:

$$A'X_2 + X_2A - X_2B_2B_2'X_2 + C_1'C_1 = 0 \tag{3-16}$$

and $W(\gamma)$ is the stabilizing solution to:

$$(A-B_2B_2'X_2)'W + W(A-B_2B_2'X_2) + WX_2^{-1}C_1'C_1X_2^{-1}W + \gamma^{-2}X_2B_1B_1'X_2 = 0 \tag{3-17}$$

whose corresponding Hamiltonian matrix is

$$H_w(\gamma) := \begin{bmatrix} (A-B_2B_2'X_2) & X_2^{-1}C_1'C_1X_2^{-1} \\ -\gamma^{-2}X_2B_1B_1'X_2 & -(A-B_2B_2'X_2)' \end{bmatrix}. \tag{3-18}$$

Since $X_\infty(\gamma)$ can be expressed in terms of $W(\gamma)$ and a constant matrix X_2, the properties of $X_\infty(\gamma)$ can be derived from those of $W(\gamma)$. Lemma 3.2 describes the γ-domain of $W(\gamma)$, and Lemma 3.3 shows some useful properties of $W(\gamma)$.

Lemma 3.2: There exists a positive number α_w such that $W(\gamma)$ is well defined everywhere on $(\alpha_w, +\infty)$, and $W(\gamma)$ does not exist on $(0, \alpha_w)$.

Proof: First of all, it is claimed that all the $j\omega$-axis eigenvalues of $H_w(\gamma)$ are identical to the $j\omega$-axis transmission zeros of the transfer function

$$\Gamma(s, \gamma) := I\gamma^2 - G_w^{\sim}(s)G_w(s) \tag{3-19}$$

where

$$G_w(s) = B_1'X_2[\, Is - (A-B_2B_2'X_2)\,]^{-1}X_2^{-1}C_1' := \hat{C}[\, Is - \hat{A}\,]^{-1}\hat{B}. \tag{3-20}$$

This can be shown by the following derivations:

$$\Gamma(s, \gamma) = I\gamma^2 - \left[\begin{array}{c|c} -\hat{A}' & -\hat{C}' \\ \hline \hat{B}' & 0 \end{array}\right]\left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & 0 \end{array}\right]$$

$$= I\gamma^2 - \left[\begin{array}{cc|c} \hat{A} & 0 & \hat{B} \\ -\hat{C}'\hat{C} & -\hat{A}' & 0 \\ \hline 0 & \hat{B}' & 0 \end{array}\right] = \left[\begin{array}{cc|c} \hat{A} & 0 & \hat{B} \\ -\hat{C}'\hat{C} & -\hat{A}' & 0 \\ \hline 0 & -\hat{B}' & I\gamma^2 \end{array}\right] := \left[\begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{C} & \tilde{D} \end{array}\right].$$

Furthermore,

$$\Gamma^{-1}(s, \gamma) = \left[\begin{array}{c|c} \tilde{A} - \tilde{B}\tilde{D}^{-1}\tilde{C} & -\tilde{B}\tilde{D}^{-1} \\ \hline \tilde{D}^{-1}\tilde{C} & \tilde{D}^{-1} \end{array}\right] = \left[\begin{array}{cc|c} \hat{A} & \gamma^2\hat{B}\hat{B}' & \gamma^2\hat{B} \\ -\hat{C}'\hat{C} & -\hat{A}' & 0 \\ \hline 0 & \gamma^2\hat{B}' & I\gamma^2 \end{array}\right] \tag{3-21}$$

By the similarity transformation

$$T=\begin{bmatrix} I\gamma^{1} & \\ & I\gamma \end{bmatrix},$$

we have

$$\Gamma^{-1}(s,\gamma)=\left[\begin{array}{cc|c} \hat{A} & \hat{B}\hat{B}' & \gamma^{1}\hat{B} \\ -\gamma^{2}\hat{C}'\hat{C} & -\hat{A}' & 0 \\ \hline 0 & \gamma^{1}\hat{B}' & I\gamma^{2} \end{array}\right]. \tag{3-22}$$

The last equation shows that { $H_w(\gamma)$, $[\gamma^1 C_1 X_2^{-1}\ \ 0]'$, $[0\ \ \gamma^1 C_1 X_2^{-1}]$, $I\gamma^2$ } is a realization of $\Gamma^{-1}(s,\gamma)$. Since $\hat{A} = (A-B_2B_2'X_2)$ is stable, by applying PBH test [19], it is clear that on the $j\omega$-axis we have

$$\operatorname{rank}\begin{bmatrix} \hat{A}-Is & \hat{B}\hat{B}' & \gamma^{-1}\hat{B} \\ -\gamma^{-2}\hat{C}'\hat{C} & -\hat{A}'-Is & 0 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} \hat{A}-Is & 0 & \gamma^{-1}\hat{B} \\ -\gamma^{-2}\hat{C}'\hat{C} & -\hat{A}'-Is & 0 \end{bmatrix} = 2n;$$

$$\operatorname{rank}\begin{bmatrix} \hat{A}-Is & \hat{B}\hat{B}' \\ -\gamma^{-2}\hat{C}'\hat{C} & -\hat{A}'-Is \\ 0 & \gamma^{-1}\hat{B}' \end{bmatrix} = \operatorname{rank}\begin{bmatrix} \hat{A}-Is & 0 \\ -\gamma^{-2}\hat{C}'\hat{C} & -\hat{A}'-Is \\ 0 & \gamma^{-1}\hat{B}' \end{bmatrix} = 2n.$$

This implies that the above realization is both controllable and observable on the $j\omega$-axis. That is, the realization does not have pole-zero cancellation on the $j\omega$-axis. Hence, the set of the eigenvalues of $H_w(\gamma)$ on the $j\omega$-axis is identical to the set of the $j\omega$-axis poles of $\Gamma^{-1}(s,\gamma)$, which in turn is identical to the set of the $j\omega$-axis transmission zeroes of $\Gamma(s,\gamma)$.

Next, we claim that if there exists a γ_1 such that $\Gamma(s,\gamma_1)$ has transmission zeros on the $j\omega$-axis, then $\Gamma(s,\gamma)$ will have transmission zeros on the $j\omega$-axis for any $\gamma \in (0, \gamma_1]$. To prove this, one just needs to note that $G_w(s)$ is a strictly proper transfer function and therefore it goes to zero, as $\omega \to +\infty$. Hence, for any $\gamma \le \gamma_1$, there always exists an ω such that $\Gamma(j\omega,\gamma)$ is singular. For instance, in Fig.3.2, we can see that if $\Gamma(s,\gamma_1)$ has a transmission zero at $j\omega_1$, then for any $\gamma_2 < \gamma_1$, there exists an ω_2 such that $j\omega_2$ is a

transmission zero of $\Gamma(s, \gamma)$.

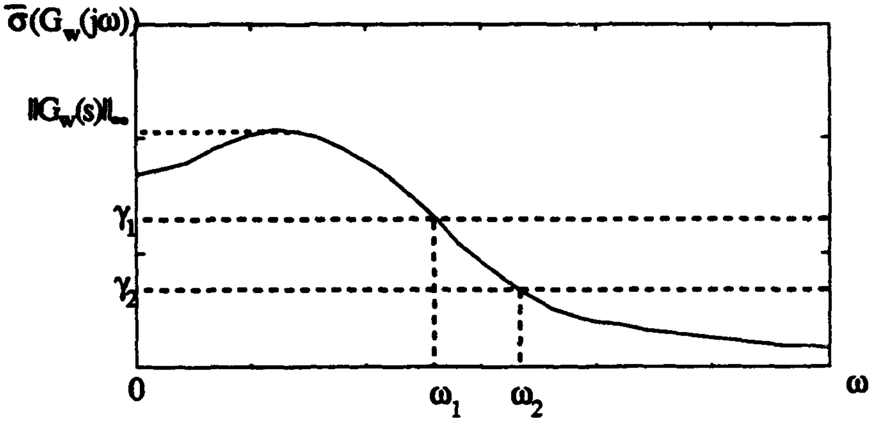


Fig. 3.2 The $j\omega$-axis transmission zeros of $\Gamma(s, \gamma)$ occurs at the frequencies such that $\gamma = \overline{\sigma}(G_w(j\omega))$.

These two claims imply the fact that if there exists a γ_1 such that $H_w(\gamma)$ has $j\omega$-axis eigenvalues, then $H_w(\gamma)$ has $j\omega$-axis eigenvalues for any γ less than γ_1. Therefore, if we define

$$\alpha_w = \inf \{\gamma: H_w(\gamma) \text{ has no } j\omega\text{-axis eigenvalues}\}, \tag{3-23}$$

then it is easy to see that $H_w(\gamma)$ has $j\omega$-axis eigenvalues for any $\gamma \in (0, \alpha_w)$ and has no $j\omega$-axis eigenvalues for any $\gamma \in (\alpha_w, +\infty)$.

Finally, since $A - B_2 B_2' X_2$ is stable and $H_w(\gamma)$ has no $j\omega$-axis eigenvalues for any $\gamma \in (\alpha_w, +\infty)$, then $W(\gamma)$, the stabilizing solution of $H_w(\gamma)$, exists for any $\gamma \in (\alpha_w, +\infty)$ [20]. Thus, we complete the proof.

From the above proof, it is easy to see that the α_w defined in (3-23) can be expressed by the H^∞ norm of a known transfer function:

$$\alpha_w = \| G_w(s) \|_\infty, \tag{3-24}$$

where $G_w(s)$ is defined in (3-20).

Lemma 3.3: On $(\alpha_w, +\infty)$,

a) $W_\gamma := dW/d\gamma \le 0$, $W_{\gamma\gamma} := d^2W/d\gamma^2 \ge 0$;

b) all eigenvalues of $W(\gamma)$ are analytic, nonincreasing functions of γ;

c) $W(\gamma) \ge 0$.

Proof: Since $W(\gamma)$ is an analytic function of γ on $(\alpha_w, +\infty)$, it is well-known [21] that, by appropriate ordering, the eigenvalues $\{\lambda_i\}$ and eigenvectors $\{v_i\}$ of $W(\gamma)$ are analytic functions of γ on $(\alpha_w, +\infty)$ such that $(I\lambda_i - W(\gamma))v_i = 0$, $i = 1, 2, \cdots, n$. Rewrite Riccati equation (3-17) as:

$$\hat{A}'W + W\hat{A} + W\hat{B}\hat{B}'W + \mu\hat{C}'\hat{C} = 0 \tag{3-25}$$

where $\mu = \gamma^2$, $\hat{A} = A - B_2B_2'X_2$, $\hat{B} = X_2^{-1}C_1'$ and $\hat{C} = B_1'X_2$. Differentiating the above equation with respect to μ, we have the following Lyapunov equation:

$$(\hat{A} + \hat{B}\hat{B}'W)'W_\mu + W_\mu(\hat{A} + \hat{B}\hat{B}'W) + \hat{C}'\hat{C} = 0 \tag{3-26}$$

where $W_\mu = \frac{dW}{d\mu}$. Because $\hat{A} + \hat{B}\hat{B}'W$ is stable, it can be inferred that $W_\mu \geq 0$ and W_μ is the unique solution. Continuously differentiating eq.(3-26) with respect to μ, we have $W_{\mu\mu} \geq 0$, where $W_{\mu\mu} = \frac{d^2W}{d\mu^2}$. By using the chain rule of differentiation, it yields

$$W_\gamma = \frac{dW}{d\gamma} = W_\mu \frac{d\mu}{d\gamma} = -2\gamma^{-3}W_\mu \leq 0 \tag{3-27}$$

and

$$W_{\gamma\gamma} = \frac{d^2W}{d\gamma^2} = 6\gamma^{-3}W_\mu + 4\gamma^{-6}W_{\mu\mu} \geq 0. \tag{3-28}$$

Now let us consider equation

$$(I\lambda - W)v = 0, \tag{3-29}$$

where λ is any eigenvalue of W and v a corresponding eigenvector. Differentiating eq.(3-29), we have

$$\dot{\lambda}(\gamma) = \frac{v'W_\gamma v}{v'v} \leq 0, \tag{3-30}$$

since $W_\gamma \leq 0$. Thus we proved part b).

Part c) is a direct consequence of part b). Since

$$H_w(\infty) = \begin{bmatrix} (A - B_2B_2'X_2) & X_2^{-1}C_1'C_1X_2^{-1} \\ 0 & -(A - B_2B_2'X_2)' \end{bmatrix},$$

the stabilizing solution $W(\infty) = 0$. Hence, it can be concluded that $W(\gamma) \geq 0$ on $(\alpha_w, +\infty)$

from the fact that all the eigenvalues of $W(\gamma)$ are nonincreasing.

In the proof of Theorem 3.1 we need to show that there exist at most n isolated values of γ such that $[X_2 - W(\gamma)]$ is singular, let $\lambda_i(\gamma)$ be the ith eigenvalue of $[X_2 - W(\gamma)]$, i =1,2,...n, where n is the dimension of $W(\gamma)$. Then we have the following lemma.

Lemma 3.4: If there is a $\gamma_1 \in (\alpha_x, +\infty)$ such that $\lambda_i(\gamma_1) = 0$, then $\dot{\lambda}_i(\gamma_1) > 0$.

Proof: Define a function on $(\alpha_x, +\infty)$ as

$$f(\gamma) := x'[X_2 - W(\gamma)]\, x, \tag{3-31}$$

where $x = v_i(\gamma_1)$ is an eigenvector corresponding to $\lambda_i(\gamma_1)$. By Lemma 3.3(a), it is easy to see that $f(\gamma)$ is a nondecreasing and concave function of γ on $(\alpha_x, +\infty)$. Furthermore, we have $f(+\infty)>0$, since $[X_2 - W(+\infty)] = X_2 > 0$ under the assumption of (C_1, A) being observable [17]. Hence, if there is a $\gamma_1 \in (\alpha_x, +\infty)$ such that $f(\gamma_1) = 0$ then $\dot{f}(\gamma_1)>0$. By this property, we can see that $\lambda_i(\gamma_1)=0$ implies $\dot{f}(\gamma_1)>0$, which in turn implies $\dot{\lambda}_i(\gamma_1)>0$.

Together with the monotonicity of $\lambda_i(\gamma)$, it comes the conclusion that $[X_2 - W(\gamma)]$ has at most n singularity points on $(\alpha_x, +\infty)$.

With these lemmas, now we are ready to prove Theorem 3.2.

The Proof of Theorem 3.2:

a) Since $X_\infty(\gamma) = X_2[\, X_2 - W(\gamma)\,]^{-1}X_2$ (from Lemma 3.1), it is obvious that:

$$\alpha_x \equiv \alpha_w \tag{3-32}$$

where α_x was defined in Theorem 3.2 and α_w was defined in Lemma 3.2. It is possible for $[X_2 - W(\gamma)]$ to lose rank at some points on $(\alpha_x, +\infty)$, therefore $X_\infty(\gamma)$ is not well defined at these points. In Lemma 3.4, we show that there are at most n such isolated points where n is the dimension of X_∞.

b) Suppose that there exists a γ_1 such that $X_\infty(\gamma_1) > 0$, and hence $[X_2 - W(\gamma_1)] > 0$. Because all the eigenvalues of $W(\gamma)$ are nonincreasing (from Lemma 3.3), $[\, X_2 - W(\gamma)\,]$ will not become singular for all $\gamma \geq \gamma_1$, which implies that $X_\infty(\gamma)$ keeps positive definite for

all $\gamma \in [\gamma_1, +\infty)$. Therefore, if we define

$$\beta_x = \inf \{\gamma : X_\infty(\gamma) \text{ is positive definite}\}, \tag{3-33}$$

then the proof is completed.

The following theorem characterizes β_x and shows how to compute it.

Theorem 3.3: Define $f(\gamma) := \lambda_{max}[W(\gamma)X_2^{-1}]$ on $(\alpha_x, +\infty)$, then

a) either $f(\beta_x) = 1$ or $\beta_x = \alpha_x$;

b) $f(\gamma)$ is a convex decreasing function of γ;

where λ_{max} denotes the maximal eigenvalue.

Proof : The proof of part a) is trivial, which is simply due to the fact that $(X_2 - W(\gamma)) > 0$ if and only if $\lambda_{max}[W(\gamma)X_2^{-1}] < 1$. To prove part b), we consider the following equation

$$[\lambda X_2 - W(\gamma)]\, u = 0. \tag{3-34}$$

Note that λ in the above equation is an eigenvalue of $W(\gamma)X_2^{-1}$. Taking derivative with respect to γ on eq.(3-34), we have

$$(\dot{\lambda} X_2 - W_\gamma)\, u + [\lambda X_2 - W(\gamma)]\, \dot{u} = 0 \tag{3-35}$$

It is easy to show that

$$\dot{\lambda}(\gamma) = \frac{u' W_\gamma u}{u' X_2 u} \leq 0 \tag{3-36}$$

on $(\alpha_x, +\infty)$, since $W_\gamma \leq 0$ from Lemma 3.3. If we replace λ in eq.(3-35) by λ_{max}, the maximal eigenvalue of $W(\gamma)X_2^{-1}$, then we have

$$\dot{u}'(\dot{\lambda}_{max} X_2 - W_\gamma)\, u = -\, \dot{u}'[\lambda_{max} X_2 - W(\gamma)]\, \dot{u} \tag{3-37}$$

where u is the eigenvalue corresponding to the λ_{max}. Note that eq.(3-35) implies

$$u'(\dot{\lambda}_{max} X_2 - W_\gamma) u = 0. \tag{3-38}$$

Taking derivative on eq.(3-38) yields

$$\dot{u}'(\dot{\lambda}_{max} X_2 - W_\gamma)\, u + u'(\dot{\lambda}_{max} X_2 - W_\gamma)\dot{u} + u'(\ddot{\lambda}_{max} X_2 - W_{\gamma\gamma})\, u = 0. \tag{3-39}$$

Combining eq.(3-37) and eq.(3-39), we have

$$\ddot{\lambda}_{max}(\gamma) = \frac{u'W_{\gamma\gamma}u + 2\dot{u}'[\lambda_{max}X_2 - W(\gamma)]\dot{u}}{u'X_2u} \quad . \tag{3-40}$$

Since $(\lambda_{max}X_2 - W(\gamma)) \geq 0$ and $W_{\gamma\gamma} \geq 0$ from Lemma 3.3, we have $\ddot{\lambda}_{max}(\gamma) \geq 0$, which, together with eq.(3-26), implies that $f(\gamma)$ is a convex decreasing function of γ on $(\alpha_x, +\infty)$.

Corollary 3.1: The derivative of $f(\gamma)$ can be computed by

$$\frac{df(\gamma)}{d\gamma} = \frac{w'W_\gamma X_2^{-1}v}{w'v}\,, \tag{3-41}$$

where w and v are right and left eigenvectors of $W(\gamma)X_2^{-1}$ corresponding to its maximal eigenvalue respectively, and W_γ can be computed from eq.(3-26) and eq.(3-27).

Fig.3.3 illustrates two possibilities mentioned in Theorem 3.3. As shown in the figure, β_x can be obtained either by finding the value of γ such that $f(\gamma)=1$ or simply by computing α_x.

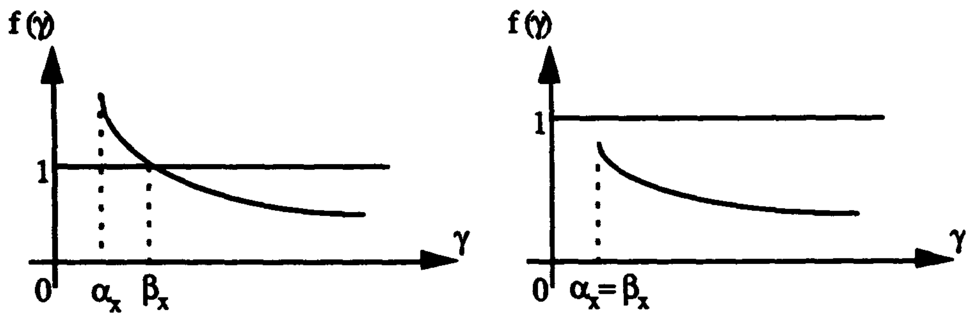


Fig. 3.3 Two possible locations of β_x.

This Corollary suggests that one can develop an algorithm to compute β_x by making use of these properties.

3.4 Properties of the H^∞ Riccati Solutions

In this section, some important properties of $X_\infty(\gamma)$ on the interval $(\beta_x, +\infty)$ will be given. We are only interested in this interval, since the H^∞ problem has no solution for any γ less than β_x (see Theorem 3.1). The properties are summarized in the following.

Theorem 3.4: On $(\beta_x, +\infty)$, $\dot{X}_\infty(\gamma) := \frac{d}{d\gamma}(X_\infty(\gamma)) \leq 0$, $\ddot{X}_\infty(\gamma) := \frac{d^2}{d\gamma^2}(X_\infty(\gamma)) \geq 0$.

Proof: Let $S(\gamma) := [\, X_2 - W(\gamma)\,]$, $\dot{S} := \frac{d}{d\gamma}\{S(\gamma)\}$ and $\dot{S}^{-1} := \frac{d}{d\gamma}\{S(\gamma)^{-1}\}$, we have

$$\dot{S} = - W_\gamma,\ \dot{S}^{-1} := - S^{-1}\dot{S}\, S^{-1} = S^{-1}W_\gamma S^{-1}.$$

From Lemma 3.1,

$$\begin{aligned} \dot{X}_\infty(\gamma) &= \frac{d}{d\gamma}\{X_2[\, X_2 - W(\gamma)\,]^{-1}X_2\} = X_2\, \dot{S}^{-1}\, X_2 \\ &= X_2\, S^{-1}W_\gamma S^{-1}\, X_2 \end{aligned} \tag{3-42}$$

This implies that $\dot{X}_\infty(\gamma) \le 0$, since $W_\gamma \le 0$ (see eq.(3-27)). Continuously taking the derivative on eq.(3-42), we have

$$\begin{aligned} \ddot{X}_\infty(\gamma) &= X_2\dot{S}^{-1}W_\gamma S^{-1}X_2 + X_2S^{-1}W_{\gamma\gamma}S^{-1}X_2 + X_2S^{-1}W_\gamma\dot{S}^{-1}X_2 \\ &= X_2S^{-1}W_\gamma S^{-1}W_\gamma S^{-1}X_2 + X_2S^{-1}W_{\gamma\gamma}S^{-1}X_2 + X_2S^{-1}W_\gamma S^{-1}W_\gamma S^{-1}X_2 \\ &= X_2S^{-1}[2W_\gamma S^{-1}W_\gamma + W_{\gamma\gamma}]S^{-1}X_2 \end{aligned} \tag{3-43}$$

This equation, together with eq.(3-28), indicates that $\ddot{X}_\infty(\gamma) \ge 0$. Thus the proof is completed.

From these inequalities, we have the following properties for $X_\infty(\gamma)$.

Theorem 3.5: On $(\beta_x, +\infty)$,

a) all eigenvalues of $X_\infty(\gamma)$ are analytic, nonincreasing functions of γ;

b) the maximal eigenvalue of $X_\infty(\gamma)$ is a nonincreasing, convex function of γ.

Proof: a) Since $X_\infty(\gamma)$ is an analytic function of γ on $(\beta_x, +\infty)$, it is well-known [21] that, by appropriate ordering, the eigenvalues $\{\lambda_i\}$ and eigenvectors $\{v_i\}$ of $X_\infty(\gamma)$ are analytic functions of γ on $(\beta_x, +\infty)$ such that

$$(I\lambda_i - X_\infty(\gamma))v_i = 0 \qquad i = 1, 2, \cdots, n. \tag{3-44}$$

To prove the monotonicity, we take derivative on eq.(3-44):

$$(I\dot{\lambda}_i - \dot{X}_\infty)v_i + (I\lambda_i - X_\infty(\gamma))\dot{v}_i = 0, \tag{3-45}$$

which implies $\dot{\lambda}_i(\gamma) = \dfrac{v_i' \dot{X}_\infty v_i}{v_i' v_i} \le 0$, since $\dot{X}_\infty \le 0$. Thus we proved part a).

b) From eq.(3-45) with $\lambda_i = \lambda_{max}$, the maximal eigenvalue of $X_\infty(\gamma)$, and $v_i = v$, the eigenvector corresponding to λ_{max}, we have the following equation:

$$\ddot{\lambda}_{max}(\gamma) = \frac{v' \ddot{X}_\infty v + 2\dot{v}' (I\lambda_{max} - X_\infty)\dot{v}}{v' v}, \tag{3-46}$$

which implies $\ddot{\lambda}_{max}(\gamma) \ge 0$, since $\ddot{X}_\infty \ge 0$ and $(I\lambda_{max} - X_\infty) \ge 0$. Because $\ddot{\lambda}_{max}(\gamma) \ge 0$ and all eigenvalues of $X_\infty(\gamma)$ under appropriate ordering are analytic and nonincreasing on $(\beta_x, +\infty)$, it is easy to see that $\lambda_{max}(\gamma)$ is convex and nonincreasing on $(\beta_x, +\infty)$, even though $\lambda_{max}(\gamma)$ may not be smooth at some isolated points. Fig. 3.4 gives an example showing the possibility.

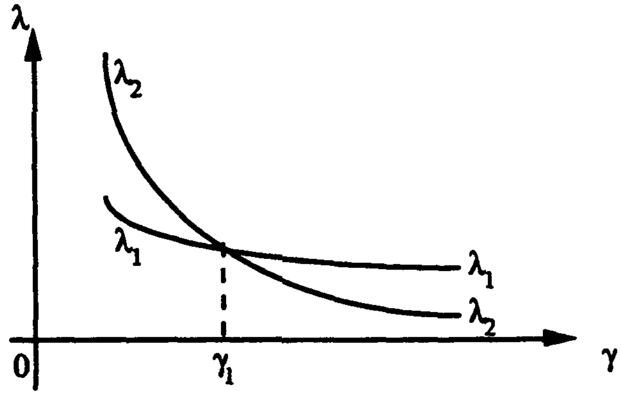


Fig. 3.4 Smoothness and convexity of the largest eigenvalue.

In the figure, both λ_1 and λ_2 are smooth functions of γ, but λ_{max} is not smooth at γ_1. However, it is still a convex function.

Since Riccati equation eq.(3-6b) is dual to eq.(3-6a), we have similar results for $Y_\infty(\gamma)$, with the following corresponding notations:

$$\alpha_y := \inf \{\gamma : \gamma \in \mathbb{R}_+ \text{ and } Y_\infty(\gamma) \text{ exists}\}$$

$$\beta_y := \inf \{\gamma : \gamma \in \mathbb{R}_+ \text{ and } Y_\infty(\gamma) \text{ is positive semidefinite}\},$$

and

$$Y_\infty(\gamma) = V \begin{bmatrix} Y(\gamma) & 0 \\ 0 & 0 \end{bmatrix} V'. \tag{3-47}$$

$Y(\gamma)$ and V are defined in a similar way as those for $X_{\infty}(\gamma)$ in eq.(3-14).

Recall that the condition, $\rho[X_{\infty}(\gamma)Y_{\infty}(\gamma)] < \gamma^2$, is required in Theorem 3.1. To investigate the properties of $X_{\infty}(\gamma)Y_{\infty}(\gamma)$, we define

$$\alpha := \max\{\alpha_x, \alpha_y\}$$
$$\beta := \max\{\beta_x, \beta_y\}.$$

Then it is easy to infer that **a)** $X_{\infty}(\gamma)Y_{\infty}(\gamma)$ exists on $(\alpha, +\infty)$ almost everywhere; **b)** $X_{\infty}(\gamma)Y_{\infty}(\gamma)$ has no negative eigenvalues on $(\beta, +\infty)$, since both $X_{\infty}(\gamma)$ and $Y_{\infty}(\gamma)$ are positive semidefinite on $(\beta, +\infty)$. Moreover, the eigenvalues of $X_{\infty}(\gamma)Y_{\infty}(\gamma)$ have the following properties.

Theorem 3.6 [16] : On $(\beta, +\infty)$,

a) all eigenvalues of $X_{\infty}Y_{\infty}$ are analytic, nonincreasing functions of γ ;

b) $\rho(\gamma):=\rho[X_{\infty}(\gamma)Y_{\infty}(\gamma)]$ is a nonincreasing, convex function of γ.

CHAPTER 4

MONOTONICITY AND CONVEXITY OF THE GD PARAMETER DEPENDENT H^∞ RICCATI SOLUTIONS

4.1 Significance of the monotonicity and convexity properties

In this chapter, we will investigate some important properties of the H^∞ Riccati solutions which play a key role in the solution of the H^∞ optimization problem. It is well known that Doyle, Glover, Khargonekar, and Francis (abbr.: DGKF) [7] presented a celebrated two-Riccati-equation type solution to a standard H^∞ control problem. Through solving two H^∞ Riccati equations, an optimal (or suboptimal) stabilizing H^∞ controller can be easily constructed. Unlike the constant Riccati solutions in Linear Quadratic Gaussian (LQG) or H^2 problem [17], the H^∞ Riccati solutions are functions of a parameter γ which is an upper bound of the optimal H^∞ norm of the closed-loop system. To construct a better suboptimal or an optimal H^∞ controller, it is necessary to search for the optimal H^∞ norm of the closed-loop system, i.e., the smallest γ such that the two H^∞ Riccati solutions $X_\infty(\gamma)$ and $Y_\infty(\gamma)$ are positive semidefinite and $\rho[X_\infty(\gamma)Y_\infty(\gamma)]$, the spectral radius of $X_\infty(\gamma)Y_\infty(\gamma)$, is less than γ^2.

Recently, efficient algorithms for computing the optimal H^∞ norm were proposed by Scherer [15] and Li and Chang [22]. Scherer considered the inverse (or pseudo inverse) of the DGKF H^∞ Riccati solutions, $X_\infty(\gamma)$ and $Y_\infty(\gamma)$, defined a new independent variable $\mu = \gamma^2$, and showed that these inverses are concave functions of μ in matrix sense on their domains of definition. Li and Chang showed that $\rho[X_\infty(\gamma)Y_\infty(\gamma)]$, the spectral radius of $X_\infty(\gamma)Y_\infty(\gamma)$, is a monotonically nonincreasing and convex function of γ. Based on these facts, quadratically convergent Newton-like algorithms were proposed to compute the optimal H^∞ norm [15,16].

Though, these monotonicity, concavity or convexity properties have only been proved for the DGKF case in which the D_{11} matrix of the generalized plant is assumed zero. This assumption can hardly be satisfied by many practical problems. In [8], Glover and Doyle (abbr.: GD) considered a more general case with D_{11} nonzero. The basic concept is the same as DGKF's, but the H^∞ Riccati equations involved are much more complicated which makes the investigation of the properties of the GD H^∞ Riccati solutions

extremely difficult.

For GD H^∞ control problem, Pandey et. al.'s hybrid gradient-bisection method [13] and Chang et. al.'s double secant and bisection method [14] were proposed for the computation of the optimal H^∞ norm. The significance of the conjecture that $\rho[X_\infty(\gamma)Y_\infty(\gamma)]$, the spectral radius of $X_\infty(\gamma)Y_\infty(\gamma)$, is a convex function of γ was mentioned in these two papers. Since there was no proof for this conjecture, bisection was used in these two algorithms as supplement to guarantee convergence.

In this chapter, we will show that the GD H^∞ Riccati solutions have the same properties possessed by the DGKF H^∞ Riccati solutions as mentioned by Li and Chang [16]. In other words, the GD H^∞ Riccati solutions $X_\infty(\gamma)$ and $Y_\infty(\gamma)$ are nonincreasing and convex functions of γ in the domain of interest and so is the spectral radius $\rho[X_\infty(\gamma)Y_\infty(\gamma)]$. Based on these properties, a quadratically convergent algorithm is proposed to compute the optimal H^∞ norm for the general H^∞ control problem considered by GD.

Section 4.2 lists the notations used in this chapter and reviews the fundamentals of the Riccati equation. The main result of the chapter is presented in Section 4.3 which shows the details of the proof of the monotonicity and convexity properties of $X_\infty(\gamma)$ and $Y_\infty(\gamma)$ in GD case. An efficient algorithm to computing the optimal H^∞ norm based on these properties will be given in Chapter 5.

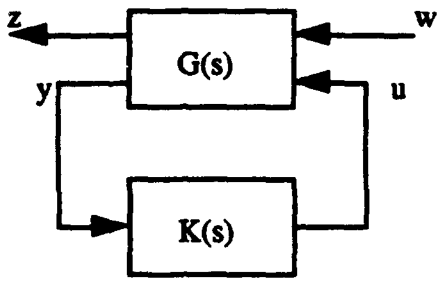
4.2 Notations and Basics of the Riccati Equation

$\mathbb{R}$	The set of real numbers.
$\mathbb{R}_+$	The set of positive real numbers.
$\mathbb{R}^{mxn}$	The set of real mxn matrices.
inf(S)	The infimum of the set S.
sup(S)	The supremum of the set S.
$\mathbb{C}^{mxn}$	The set of nxm matrices whose entries are complex numbers.
I	Identity matrix.
I_n	n-dimensional identity matrix.

0	Zero or zero matrix.
O_n	n-dimensional zero matrix.
A^T	Transpose of a matrix A.
A^*	Conjugate transpose of a matrix A.
$A \geq 0$	A is a positive semi-definite matrix.
$A > 0$	A is a positive definite matrix.
$\lambda(A)$	An arbitrary eigenvalue of A.
$\lambda_i(A)$	The ith largest eigenvalue of A.
$\lambda_{max}(A)$	The maximal eigenvalue of A.
$\lambda_{min}(A)$	The minimal eigenvalue of A.
$\rho(A)$	The spectral radius of A.
$\sigma_{max}(A)$	The maximal singular value of A.
$A(\gamma)$	The matrix A is a function of γ.
H_∞	The Hamiltonian matrix shown in (2-5a).
J_∞	The Hamiltonian matrix shown in (2-5b).
$X_\infty(\gamma)$	The stabilizing solution to the Riccati equation corresponding to H_∞.
$Y_\infty(\gamma)$	The stabilizing solution to the Riccati equation corresponding to J_∞.
μ	The inverse of γ square.
μ_i	$\mu_i := \frac{1}{\gamma^2 - \lambda_i}$, where λ_i is the ith eigenvalue of $D_{11}^T D_\perp D_\perp^T D_{11}$.
$H(\mu)$	Defined as $H_\infty\big\|_{\gamma=\mu^{-1/2}}$.

$J(\mu)$	Defined as $J_\infty\big|_{\gamma=\mu^{-1/2}}$.
$X(\mu)$	The antistabilizing solution to the Riccati equation corresponding to the following Hamiltonian matrix $$-\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} H(\mu) \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}.$$
$\dfrac{\partial X}{\partial \mu_i}$	The first order partial derivative of X with respect to μ_i, i. e., $\dot{X}_{\mu i}$.
$\dfrac{\partial^2 X}{\partial \mu_i \partial \mu_j}$	The second order partial derivative of X with respect to μ_i and μ_j, i.e., $\ddot{X}_{\mu i \mu j}$.
$Y(\mu)$	The antistabilizing solution to the Riccati equation corresponding to the following Hamiltonian matrix $$-\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} J(\mu) \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$$
$\tilde{X}(\gamma)$	Defined as $X(\mu)\big|_{\mu=\gamma^{-2}}$.
$\tilde{Y}(\gamma)$	Defined as $Y(\mu)\big|_{\mu=\gamma^{-2}}$.
α_x	Inf $\{\gamma : \gamma \in \mathbb{R}_+, \gamma > \overline{\sigma}(D_\perp{}^T D_{11})$ and X_∞ exists $\}$.
α_y	Inf $\{\gamma : \gamma \in \mathbb{R}_+, \gamma > \overline{\sigma}(\tilde{D}_\perp D_{11}{}^T)$ and Y_∞ exists $\}$.
β_x	Inf $\{\gamma : \gamma \in \mathbb{R}_+, \gamma \geq \alpha_x$ and $\tilde{X}(\gamma)$ is positive definite$\}$.
β_y	Inf $\{\gamma : \gamma \in \mathbb{R}_+, \gamma \geq \alpha_y$ and $\tilde{Y}(\gamma)$ is positive definite$\}$.
$\left[\begin{array}{c\|c} A & B \\ \hline C & D \end{array}\right]$	A state space realization of the transfer matrix D+C (sI - A) B.
α	$\max\{\alpha_x, \alpha_y\}$.

β	$\max\{\beta_x, \beta_y\}$.
$\rho(\gamma)$	Is defined as $\rho[X_\infty(\gamma)Y_\infty(\gamma)]$ on $(\beta, +\infty)$.
$\lVert G(s) \rVert_\infty$	The H^∞ norm of $G(s)$.
$\mathcal{F}_\ell(G,K)$	The lower linear fractional transformation which stands for the closed loop transfer matrix from w to z as shown in the following



Let A, Q, R be real n by n matrices with Q and R symmetric. The algebraic Riccati equation and its corresponding Hamiltonian matrix are shown as follows:

$$A^T X + X A + X R X - Q = 0 \tag{4-1}$$

$$H := \begin{bmatrix} A & R \\ Q & -A^T \end{bmatrix}. \tag{4-2}$$

Assume H has no eigenvalue on the imaginary axis. Then finding a basis for the invariant subspace corresponding to the eigenvalues of H in LHP, i.e., the open left half complex plane, and using the basis vectors to form a matrix, we obtain

$$X_-(H) := \mathrm{Im}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

where $X_1, X_2 \in \mathbb{R}^{n\times n}$. If the two subspaces

$$X_-(H), \mathrm{Im}\begin{bmatrix} 0 \\ I \end{bmatrix}$$

are complementary, i.e., X_1 is invertible, then the stabilizing solution to the Riccati equation is $X = X_2 X_1^{-1}$. X is uniquely determined by H, i.e., there is an operator denoted Ric which maps H to X. Thus, X = Ric(H) and the domain of Ric is denoted by dom(

Ric). Furthermore, if X exists and is nonsingular, then X_2 is also invertible.

Consider the system

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = C(sI - A)^{-1}B + D$$

with (A, B) stabilizable. The following two fundamental lemmas of the Riccati equation will be employed in the proof of the main results in Section 4.3.

Lemma 4.1 Suppose H has the form

$$H = \begin{bmatrix} A & -BB^T \\ -C^TC & -A^T \end{bmatrix} \tag{4-3}$$

Then the stabilizing solution X to the corresponding Riccati equation exists and is nonnegative semi-definite. The null space of X, i.e., ker (X), is a subset of the stable unobservable subspace.

Stable unobservable subspace means the intersection of the stable invariant subspace of A with the unobservable subspace of (A, C). Note that a detectable (C, -A) implies that X is positive definite. On the other hand, if (C, -A) is not detectable, then A has stable modes that are not observable from C. Assume that there is a similarity transformation U such that

$$U^T \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] U = \left[\begin{array}{cc|c} A_{11} & 0 & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & 0 & D \end{array}\right]$$

with A_{22} stable and $(C_1, -A_{11})$ detectable, let X_1 be the stabilizing solution to the following Riccati equation

$$A_{11}^T X_1 + X_1 A_{11} - X_1 B_1 B_1^T X_1 + C_1^T C_1 = 0,$$

then it can be shown that $X = U \begin{bmatrix} X_1 & 0 \\ 0 & 0 \end{bmatrix} U^T$ is a stabilizing solution for the Riccati equation corresponding to (4-3).

Lemma 4.2 Assume the stabilizing solution X to (4-1) is positive definite and $\hat{X}$ is the inverse of X, then $\hat{X}$ is the antistabilizing solution to

$$A\hat{X} + \hat{X}A^T - \hat{X}Q\hat{X} + R = 0 \qquad (4\text{-}4)$$

i.e., the eigenvalues of $A-\hat{X}Q$ are in RHP, i.e., the open right half complex plane.

Proof: The Hamiltonian matrix associated with eq. (4-4) is

$$\hat{H} = \begin{bmatrix} A^T & -Q \\ -R & -A \end{bmatrix}.$$

It is easy to see that

$$-\hat{H} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} H \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}.$$

If the stable invariant subspace of H is $\mathrm{Im}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$, then that of $-\hat{H}$ is

$$\mathrm{Im}\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \mathrm{Im}\begin{bmatrix} X_2 \\ X_1 \end{bmatrix}$$

and hence $\mathrm{Im}\begin{bmatrix} X_2 \\ X_1 \end{bmatrix}$ is the antistable invariant subspace of $\hat{H}$. That is, $\hat{X}$ is the antistabilizing solution to (4-4). Q.E.D.

4.3 Properties of the γ-Dependent H^∞ Riccati Solutions

The Riccati equations associated with the Hamiltonian matrices (2-5a) and (2-5b) are dual to each other, so we will only concentrate on one of them, say, eq.(2-5a). To investigate the properties of Riccati solution of eq.(2-5a), we first assume that $(D_\perp^T C_1, -A+BR^{-1}D_{1\bullet}^T C_1)$ is detectable. This assumption will be removed later in this section. With the assumption it is obvious that $X_\infty = \mathrm{Ric}(H_\infty)$ is invertible.

Although our objective is to show that X_∞ and Y_∞ are nonincreasing convex functions of γ, we will consider their inverses first since the inverses, as shown later, are analytic functions of γ.

In eq. (2-5a), the Hamiltonian matrix H_∞ depends on the inverse of R which is a

function of γ. The parameter γ appears almost everywhere in the entries of R^{-1} and the matrix H_∞ could be a messy function of γ. To avoid this messiness, the following lemma is employed to confine the influence of γ in a diagonal matrix.

Lemma 4.3 The matrix R in (2-5d) can be expressed as $-T\,\Delta(\gamma)^{-1}\,T^T$, where $\Delta(\gamma)$ is a diagonal matrix.

Proof:

By $D_{12}^T D_{12} = I$, R can be decomposed as

$$R = \begin{bmatrix} D_{11}^T D_{11} - \gamma^2 I_{m1} & D_{11}^T D_{12} \\ D_{12}^T D_{11} & I \end{bmatrix}$$

$$= \begin{bmatrix} I_{m1} & D_{11}^T D_{12} \\ 0 & I \end{bmatrix} \begin{bmatrix} (D_{11}^T D_\perp D_\perp^T D_{11} - \gamma^2 I_{m1}) & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} I_{m1} & 0 \\ D_{12}^T D_{11} & I \end{bmatrix} \tag{4-5}$$

Define $E = \begin{bmatrix} E_I & 0 \\ 0 & I \end{bmatrix}$ where E_I is the orthogonal matrix in the singular value decomposition of $D_\perp^T D_{11}$, i.e., $D_{11}^T D_\perp D_\perp^T D_{11} = E_I\,\mathrm{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_{m1}\}\,E_I^T$, and let

$$T := \begin{bmatrix} I_{m1} & D_{11}^T D_{12} \\ 0 & I \end{bmatrix} E,$$

and

$$\Delta(\gamma) := \mathrm{diag}\{\mu_1^{-1}, \mu_2^{-1}, \ldots, \mu_{m1}^{-1}, -1, \ldots, -1\},$$

where $\mu_i := \dfrac{1}{\gamma^2 - \lambda_i}$ $\qquad i = 1, 2, \ldots, m_1$,

then it is trivial to show that

$$R = -T\,\Delta(\gamma)^{-1}\,T^T. \qquad \text{Q.E.D.}$$

This lemma shows that the effect of γ is only on the eigenvalues of R. This fact plays a key role in reducing the complexity of the proof that the X_∞ is a convex function of γ.

In order to specify the domain of the mapping $X_\infty(\gamma)$, we define

$$\alpha_x := \inf \{\gamma : \gamma \in \mathbb{R}_+, \gamma > \bar{\sigma}(D_\perp^T D_{11}) \text{ and } X_\infty \text{ exists}\}.$$

To simplify the proof, let $\mu := \gamma^{-2}$ and define

$$D_I := \begin{bmatrix} I_{m1} \\ -D_{12}^T D_{11} \end{bmatrix}, \quad D_{II} := \begin{bmatrix} 0 \\ I \end{bmatrix}, \tag{4-6}$$

$$\begin{bmatrix} T_1 \\ -T_2 \end{bmatrix} = \begin{bmatrix} B \\ -C_1^T D_{1\bullet} \end{bmatrix} \begin{bmatrix} D_I & D_{II} \end{bmatrix}, \tag{4-7}$$

and

$$M := \begin{bmatrix} (\mu I_{m1} - D_{11}^T D_\perp D_\perp^T D_{11})^{-1} & 0 \\ 0 & -I \end{bmatrix}.$$

Then the Hamiltonian matrix H_∞ in (2-5a) can be written as

$$H(\mu) := \begin{bmatrix} (A+T_1 M T_2^T) & T_1 M T_1^T \\ -(C_1^T C_1 + T_2 M T_2^T) & -(A+T_1 M T_2^T)^T \end{bmatrix}. \tag{4-8}$$

Consider the following Riccati equation

$$X(A+T_1M T_2^T)^T + (A+T_1M T_2^T)X + T_1M T_1^T + X(C_1^T C_1+T_2M T_2^T)X = 0 \tag{4-9}$$

whose associated Hamiltonian matrix is

$$-\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} H(\mu) \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}.$$

It is easy to see that $X(\mu)$ exists only on the interval $(0, \alpha_x^{-2})$. In the proof of the following theorem, the vector vec X, associated with the matrix X, is defined as follows.

Definition 4.1: With each matrix $X = [x_{ij}] \in \mathbb{R}^{n\times n}$, we associate the vector vec $X \in \mathbb{R}^{nn}$ defined by

$$\text{vec } X = [x_{11}, \ldots, x_{n1}, x_{12}, \ldots\ x_{n2}, \ldots, x_{1n}, \ldots, x_{nn}]^T \tag{4-10}$$

Theorem 4.1: On $(0, \alpha_x^{-2})$, $X(\mu)$ is analytic and satisfies

$$\frac{dX}{d\mu} \leq 0, \tag{4-11}$$

i.e., $X(\mu)$ is an analytic nonincreasing function.

Proof:

To prove the analyticity of X, we define

$F(X,\mu) := X(A+T_1M\,T_2^T)^T + (A+T_1M\,T_2^T)X + T_1M\,T_1^T + X(C_1^T C_1+T_2M\,T_2^T)X$

where $X \in \mathbf{X}$

$$\mathbf{X} := \{X \in \mathbb{R}^{nxn} \mid X \text{ is positive definite and } (A+T_1M\,T_2^T)+X(C_1^T C_1+T_2M\,T_2^T) \text{ is antistable}\}$$

From the above definition, we have

$$\begin{aligned} F(X+\delta X,\mu) - F(X,\mu) \\ &= \delta X\,[(A+T_1M\,T_2^T)+X(C_1^T C_1+T_2M\,T_2^T)]^T \\ &\quad + [(A+T_1M\,T_2^T)+X(C_1^T C_1+T_2M\,T_2^T)]\,\delta X \\ &\quad + \delta X\,(C_1^T C_1+T_2M\,T_2^T)\,\delta X \end{aligned} \tag{4-12}$$

where δX is a small perturbation on X. Let

$$\tilde{A} := [(A+T_1M\,T_2^T)+X(C_1^T C_1+T_2M\,T_2^T)],$$

then the partial Fréchet differential [23] of F(., .) with respect to X is

$$\partial F(X,\mu) = \partial X\,\tilde{A}^T + \tilde{A}\,\partial X \tag{4-13}$$

and it can be regarded as a linear map. The map

$$\partial X \text{ a } \partial X\,\tilde{A}^T + \tilde{A}\,\partial X \tag{4-14}$$

is nonsingular because

$$\text{vec}\,\partial F(X,\mu) = \text{vec}\,\partial X\,\tilde{A}^T + \text{vec}\,\tilde{A}\,\partial X$$

or equivalently,

$$\text{vec}\,\partial F(X,\mu) = [I \otimes \tilde{A} + \tilde{A} \otimes I]\,\text{vec}\,\partial X \tag{4-15}$$

where $\otimes$ is the Kronecker product. $[I \otimes \tilde{A} + \tilde{A} \otimes I]$ is nonsingular because of the antistabilizability of X. Consequently, $\partial F(X,\mu)$ and ∂X are uniquely determined by each

other.

It is clear that F(., .) is an analytic function which maps (X, μ) to F(X, μ) on the domain $\mathbf{X} \times \mathbf{L}$ where $\mathbf{L} = (0, \alpha_x^{-2})$. From the implicit function theorem [23] and analyticity of F(., .), there exists a neighborhood, (μ-ε, μ+ε) with $\varepsilon > 0$, such that the map μ a X is analytic. Hence, X is an analytic function of μ. Next, we will show that X is a nonincreasing function of μ. Let

$$\dot{X}_{\mu i} = \frac{\partial X}{\partial \mu_i},$$

$$\dot{M}_{\mu i} = \frac{\partial M}{\partial \mu_i} \qquad i = 1, 2, \ldots, m_1$$

Differentiating eq. (4-8) with respect to μ_i gives

$$\dot{X}_{\mu i}\tilde{A}^T + \tilde{A}\dot{X}_{\mu i} + (XT_2 + T_1)\dot{M}_{\mu i}(XT_2 + T_1)^T = 0 \quad i = 1, 2, \ldots, m_1 \tag{4-16}$$

Recalling that $M = E\Delta E^T$, we have

$$\dot{M}_{\mu i} = E\frac{\partial \Delta}{\partial \mu_i}E^T = E\,\mathrm{diag}(0, \ldots, 0, \underbrace{1}_{\text{i-th}}, 0, \ldots, 0)\,E^T$$

which is positive semi-definite. Since X is the antistabilizing solution to (4-9) i.e., $\tilde{A}$ is antistable, eq. (4-16) implies that

$$\dot{X}_{\mu i} \le 0 \tag{4-17}$$

and therefore

$$\frac{dX}{d\mu} = \sum_{i=1}^{m1} \dot{X}_{\mu i} \frac{1}{(1-\lambda_i \mu)^2} \le 0 \qquad \text{Q.E.D.}$$

Based on Theorem 4.1, it is known that the second order or higher order derivatives of $X(\mu)$ with respect to μ exist on $(0, \alpha_x^{-2})$. To prove that $X(\mu)$ is a concave function in Theorem 4.2, we need the following lemma.

Lemma 4.4: Let

$$G := E\,\mathrm{diag}\{\mu^{-1}\mu_1, \mu^{-1}\mu_2, \ldots, \mu^{-1}\mu_{m1}, 0, \ldots, 0\}\,E^T \tag{4-18}$$

and

$$W := \begin{bmatrix} C_1^T D_\perp D_\perp^T C_1 + \mu T_2 G T_2^T & T_2 G^2 \\ G^2 T_2^T & \mu^{-1} G^3 - \mu^{-1} G^2 \end{bmatrix},$$

then on $(0, \alpha_x^{-2})$ the matrix W is positive semi-definite.

Proof: G can be rewritten as

$$G = \mathrm{diag}\{\mu^{-1}(\mu^{-1} I_{m1} - D_{11}^T D_\perp D_\perp^T D_{11})^{-1}, 0, \ldots, 0\}.$$

Let

$$W_I := \mu^{-1}(\mu^{-1} I_{m1} - D_{11}^T D_\perp D_\perp^T D_{11})^{-1}$$

and

$$W_{II} := E\,\mathrm{diag}\{\frac{\lambda_1}{(1-\lambda_1\mu)^3}, \frac{\lambda_2}{(1-\lambda_2\mu)^3}, \ldots, \frac{\lambda_{m1}}{(1-\lambda_{m1}\mu)^3}\}E^T,$$

then we have

$$\mu^{-1}(G^3 - G^2) = \mathrm{diag}\{W_{II}, 0, \ldots, 0\}.$$

Since

$$T_2 G^2 = [C_1^T D_\perp D_\perp^T D_{11} W_I^2 \quad 0],$$

W can be written as

$$W := \begin{bmatrix} W_s & 0 \\ 0 & 0 \end{bmatrix}$$

where

$$W_s = \begin{bmatrix} C_1^T D_\perp D_\perp^T C_1 + \mu T_2 G T_2^T & C_1^T D_\perp D_\perp^T D_{11} W_I^2 \\ W_I^2 D_{11}^T D_\perp D_\perp^T C_1 & W_{II} \end{bmatrix}$$

It is obvious that W is positive semi-definite if and only if so is W_s. In the following, we will show that W_s is positive semi-definite. W_s can be decomposed as

$$W_s=\begin{bmatrix} C_1^TD_\perp & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} I+\mu D_\perp^T D_{11} W_I D_{11}^T D_\perp & D_\perp^T D_{11} W_I^2 \\ W_I^2 D_{11}^T D_\perp & W_{II} \end{bmatrix}\begin{bmatrix} D_\perp^T C_1 & 0 \\ 0 & I \end{bmatrix}.$$

The matrix $Q := I + \mu D_\perp^T D_{11} W_I D_{11}^T D_\perp$ is positive definite since W_I is. The middle matrix on the right hand side of the above equation can be further decomposed as

$$\begin{bmatrix} Q & D_\perp^T D_{11} W_I^2 \\ W_I^2 D_{11}^T D_\perp & W_{II} \end{bmatrix}=\begin{bmatrix} I & 0 \\ W_I^2 D_{11}^T D_\perp Q^{-1} & I \end{bmatrix}\cdot$$

$$\begin{bmatrix} Q & 0 \\ 0 & W_{II}- W_I^2 D_{11}^T D_\perp Q^{-1} D_\perp^T D_{11} W_I^2 \end{bmatrix}\begin{bmatrix} I & 0 \\ W_I^2 D_{11}^T D_\perp Q^{-1} & I \end{bmatrix}^T$$

Now, the proof boils down to checking that $W_{II} - W_I^2 D_{11}^T D_\perp Q^{-1} D_\perp^T D_{11} W_I^2$ is positive semi-definite. Based on the facts shown as follows,

$$\begin{aligned} Q^{-1} &= (I + \mu D_\perp^T D_{11} W_I D_{11}^T D_\perp)^{-1} \\ &= \Gamma^{-1} - \Gamma^{-1} D_\perp^T D_{11} [D_{11}^T D_\perp \Gamma^{-1} D_\perp^T D_{11} + (\mu W_I)^{-1}]^{-1} D_{11}^T D_\perp \Gamma^{-1} \\ &= I - D_\perp^T D_{11} [D_{11}^T D_\perp D_\perp^T D_{11} + (\mu\mu^{-1}(\mu^{-1} I_{m1} - D_{11}^T D_\perp D_\perp^T D_{11})^{-1})^{-1}]^{-1} \\ &\quad \cdot D_{11}^T D_\perp \\ &= I - D_\perp^T D_{11} [D_{11}^T D_\perp D_\perp^T D_{11} + \mu^{-1} I_{m1} - D_{11}^T D_\perp D_\perp^T D_{11}]^{-1} D_{11}^T D_\perp \\ &= I - \mu^{-1} D_\perp^T D_{11} D_{11}^T D_\perp \end{aligned}$$

and

$$\begin{aligned} D_{11}^T D_\perp Q^{-1} D_\perp^T D_{11} &= D_{11}^T D_\perp (I - \mu^{-1} D_\perp^T D_{11} D_{11}^T D_\perp) D_\perp^T D_{11} \\ &= D_{11}^T D_\perp D_\perp^T D_{11} - (D_{11}^T D_\perp D_\perp^T D_{11})\, \mu\, (D_{11}^T D_\perp D_\perp^T D_{11}) \\ &= E_I \operatorname{diag}\{\lambda_1(1-\lambda_1\mu), \lambda_2(1-\lambda_2\mu), \ldots, \lambda_{m1}(1-\lambda_{m1}\mu)\} E_I^T, \end{aligned}$$

it is easy to show

$$W_{II} - W_I^2 D_{11}^T D_\perp Q^{-1} D_\perp^T D_{11} W_I^2$$

$$= E_I \operatorname{diag}\Big\{\frac{\lambda_1}{(1-\lambda_1\mu)^3} - \frac{1}{(1-\lambda_1\mu)^2}\lambda_1(1-\lambda_1\mu)\frac{1}{(1-\lambda_1\mu)^2}, \ldots,$$

$$\frac{\lambda_{m1}}{(1-\lambda_{m1}\mu)^3} - \frac{1}{(1-\lambda_{m1}\mu)^2}\lambda_{m1}(1-\lambda_{m1}\mu)\frac{1}{(1-\lambda_{m1}\mu)^2}\}E_I$$

is a zero matrix and hence

$$\begin{bmatrix} Q & D_\perp^T D_{11} W_I^2 \\ W_I^2 D_{11}^T D_\perp & W_{II} \end{bmatrix}$$

is positive semi-definite. Consequently, W_s is positive semi-definite and so is W. Q.E.D.

Based on the fact that $X(\mu)$ is an analytic nonincreasing function, as shown in Theorem 4.1, the second order derivative of $X(\mu)$ exists. Furthermore, we also found that $X(\mu)$ has the following important property.

Theorem 4.2: On $(0, \alpha_x^{-2})$, $X(\mu)$ is a concave function, that is,

$$\frac{d^2X}{d\mu^2} \le 0. \tag{4-19}$$

Proof:

Taking partial derivative of (4-16) with respect to μ_j, we have

$$\ddot{X}_{\mu i\mu j}\bar{A}^T + \bar{A}\ddot{X}_{\mu i\mu j} + \dot{X}_{\mu i}T_2\dot{M}_{\mu j}(XT_2+T_1)^T + \dot{X}_{\mu i}(C_1^T C_1+T_2 M T_2^T)\dot{X}_{\mu j}$$
$$+ (XT_2+T_1)\dot{M}_{\mu j}T_2^T\dot{X}_{\mu i} + \dot{X}_{\mu j}(C_1^T C_1 + T_2 M T_2^T)\dot{X}_{\mu i} + \dot{X}_{\mu j}T_2\dot{M}_{\mu i}(XT_2+T_1)^T$$
$$+ (XT_2+T_1)\dot{M}_{\mu i}T_2^T\dot{X}_{\mu j} = 0$$
$$i=1, 2, ..., m_1, \qquad j= 1, 2, ..., m_1 \tag{4-20}$$

For notation simplicity, we define

$$\ddot{X}_I = \sum_{i=1}^{m1}\sum_{j=1}^{m1}(\ddot{X}_{\mu i\mu j}\frac{1}{(1-\lambda_i\mu)^2}\frac{1}{(1-\lambda_j\mu)^2})$$

and

$$\dot{X}_I = \sum_{i=1}^{m1}\dot{X}_{\mu i}\frac{1}{(1-\lambda_i\mu)^2}$$

From eq. (4-18), it is obvious that

$$G^{K} = \sum_{i=1}^{m1} \dot{M}_{\mu i} \frac{1}{(1-\lambda_i \mu)^{K}} \quad (4\text{-}21)$$

Now, the sum

$$\sum_{i=1}^{m1} \sum_{j=1}^{m1} \left(\frac{1}{(1-\lambda_i \mu)^2 (1-\lambda_j \mu)^2} (4\text{-}20) \right)$$

can be simplified as

$$\ddot{X}_I \tilde{A}^T + \tilde{A}\ddot{X}_I + 2\dot{X}_I T_2 G^2 (XT_2+T_1)^T + 2(XT_2+T_1) G^2 T_2^T \dot{X}_I$$
$$+ 2\dot{X}_I (C_1^T C_1 + T_2 M T_2^T) \dot{X}_I = 0 \quad (4\text{-}22)$$

Define

$$\dot{X}_{II} = \sum_{i=1}^{m1} \dot{X}_{\mu i} \frac{2\lambda_i}{(1-\lambda_i \mu)^3}$$

then the sum

$$\sum_{i=1}^{m1} \frac{2\lambda_i}{(1-\lambda_i \mu)^2} (4\text{-}16)$$

leads to

$$\dot{X}_{II} \tilde{A}^T + \tilde{A}\dot{X}_{II} + (XT_2+T_1) \left(\sum_{i=1}^{m1} \dot{M}_{\mu i} \frac{2\lambda_i}{(1-\lambda_i \mu)^3} \right) (XT_2+T_1)^T = 0 \quad (4\text{-}23)$$

Adding eq. (4-23) to eq. (4-22) gives

$$(\ddot{X}_I + \dot{X}_{II}) \tilde{A}^T + \tilde{A}(\ddot{X}_I + \dot{X}_{II}) + 2\dot{X}_I T_2 G^2 (XT_2 + T_1) + 2(XT_2 + T_1) G^2 T_2^T \dot{X}_I$$
$$+ 2\dot{X}_I (C_1^T C_1 + T_2 M T_2^T) \dot{X}_I + 2(XT_2+T_1) \left(\sum_{i=1}^{m1} \dot{M}_{\mu i} \frac{\lambda_i}{(1-\lambda_i \mu)^3} \right) (XT_2 + T_1)^T = 0$$

(4-24)

Based on the fact that

$$D_1 \cdot D_I = D_{11} - D_{12} D_{12}^T D_{11} = D_\perp D_\perp^T D_{11}, \; D_1 \cdot D_{II} = D_{12}$$

and

$$M = \mu G + \mathrm{diag}\{O_{m1}, -I\}$$

where O_{m1} is the m_1 by m_1 zero matrix, we have

$$T_2 = [\, C_1^T D_\perp D_\perp^T D_{11} \quad C_1^T D_{12}]$$

and

$$C_1^T C_1 + T_2\, \mathrm{diag}\{O_{m1}, -I\}\, T_2^T = C_1^T D_\perp D_\perp^T C_1.$$

Then eq. (4-24) becomes

$$\begin{aligned}(\ddot{X}_I + \dot{X}_{II})\tilde{A}^T + \tilde{A}(\ddot{X}_I + \dot{X}_{II}) + 2\dot{X}_I T_2 G^2 (XT_2 + T_1) + 2(XT_2 + T_1) G^2 T_2^T \dot{X}_I \\ + 2\dot{X}_I (C_1^T D_\perp D_\perp^T C_1)\, \dot{X}_I + 2\mu \dot{X}_I T_2 G\, T_2^T \dot{X}_I \\ + 2(XT_2 + T_1)\Big(\sum_{i=1}^{m1} \dot{M}_{\mu i} \frac{\lambda_i}{(1-\lambda_i \mu)^3}\Big)(XT_2 + T_1)^T = 0 \end{aligned} \qquad (4\text{-}25)$$

With

$$\sum_{i=1}^{m1} \dot{M}_{\mu i} \frac{\lambda_i}{(1-\lambda_i \mu)^3} = \sum_{i=1}^{m1} \dot{M}_{\mu i}\Big(\frac{1}{(1-\lambda_i \mu)^3} - \frac{1}{(1-\lambda_i \mu)^2}\Big)\mu^{-1} = \mu^{-1}(G^3 - G^2),$$

eq. (4-25) becomes

$$\begin{aligned}(\ddot{X}_I + \dot{X}_{II})\tilde{A}^T + \tilde{A}(\ddot{X}_I + \dot{X}_{II}) + 2\dot{X}_I T_2 G^2 (XT_2 + T_1) + 2(XT_2 + T_1) G^2 T_2^T \dot{X}_I \\ + 2\dot{X}_I (C_1^T D_\perp D_\perp^T C_1)\, \dot{X}_I + 2\mu \dot{X}_I T_2 G\, T_2^T \dot{X}_I \\ + 2(XT_2 + T_1)(\mu^{-1} G^3 - \mu^{-1} G^2)(XT_2 + T_1)^T = 0 \end{aligned}$$

This suggests that

$$(\ddot{X}_I+\dot{X}_{II})\tilde{A}^T + \tilde{A}(\ddot{X}_I+\dot{X}_{II}) + 2\dot{X}_I(C_1^T D_\perp D_\perp^T C_1)\dot{X}_I$$

$$- 2(XT_2+T_1)(\mu^{-1}G^2)(XT_2+T_1)^T$$

$$+ 2\left[\dot{X}_I T_2+\mu^{-1}(XT_2+T_1)G\right](\mu G)\left[\dot{X}_I T_2+\mu^{-1}(XT_2+T_1)G\right]^T = 0 \qquad (4\text{-}26)$$

Note that

$$2\dot{X}_I(C_1^T D_\perp D_\perp^T C_1)\dot{X}_I - 2(XT_2+T_1)(\mu^{-1}G^2)(XT_2+T_1)^T$$

$$+ 2\left[\dot{X}_I T_2+\mu^{-1}(XT_2+T_1)G\right](\mu G)\left[\dot{X}_I T_2+\mu^{-1}(XT_2+T_1)G\right]^T$$

$$= V\,\mathrm{diag}\{D_{11}^T D_\perp D_\perp^T D_{11}, -\mu I, \mu G\}\, V^T,$$

where

$$V := \left[\dot{X}_I \quad \mu^{-1}(XT_2+T_1) \quad \dot{X}_I T_2+\mu^{-1}(XT_2+T_1)G\right].$$

Now,

$$V\,\mathrm{diag}\{D_{11}^T D_\perp D_\perp^T D_{11}, -\mu I, \mu G\}\, V^T$$

$$= \tilde{V}\begin{bmatrix} C_1^T D_\perp D_\perp^T C_1 + \mu T_2 G T_2^T & T_2 G^2 \\ G^2 T_2^T & \mu^{-1}G^3 - \mu^{-1}G^2 \end{bmatrix}\tilde{V}^T$$

$$= \tilde{V}\, W\, \tilde{V}^T$$

where

$$\tilde{V} := \left[\dot{X}_I \quad \mu^{-1}(XT_2+T_1)G\right].$$

Since W is positive semi-definite from Lemma 4.4,

$$V\,\mathrm{diag}\{D_{11}^T D_\perp D_\perp^T D_{11}, -\mu I, \mu G\}\, V^T$$

is also positive semi-definite.

In (4-26), antistable $\tilde{A}$ and positive semi-definite $V\,\mathrm{diag}\{D_{11}^T D_\perp D_\perp^T D_{11}, -\mu I, \mu G\}\, V^T$ imply that

$$(\ddot{X}_I+\dot{X}_{II}) \le 0 \qquad (4\text{-}27)$$

Based on the fact:

$$\frac{d^2X}{d\mu^2} = \sum_{i=1}^{m1}\sum_{j=1}^{m1}(\ddot{X}_{\mu i\mu j}\frac{1}{(1-\lambda_i\mu)^2}\frac{1}{(1-\lambda_j\mu)^2}) + \sum_{i=1}^{m1}\dot{X}_{\mu i}\frac{2\lambda_i}{(1-\lambda_i\mu)^3} = \ddot{X}_I + \dot{X}_{II}$$

and (4-27), it is clear that

$$\frac{d^2X}{d\mu^2} \le 0. \qquad \text{Q.E.D.}$$

Define

$$\tilde{X}(\gamma) := X(\mu)\big|_{\mu=\gamma^{-2}},$$

then we have the following corollary which is a direct result from Theorem 4.1 and Theorem 4.2.

Corollary 4.1: $\tilde{X}(\gamma)$ is a well-defined function on $(\alpha_x, +\infty)$. Moreover, $\tilde{X}(\gamma)$ is analytic and satisfies

$$\frac{d\tilde{X}(\gamma)}{d\gamma} \ge 0 \qquad (4\text{-}28)$$

and

$$\frac{d^2\tilde{X}(\gamma)}{d\gamma^2} \le 0 \qquad (4\text{-}29)$$

on $(\alpha_x, +\infty)$.

Proof:

Since $\tilde{X}(\gamma)$ is a compound function consisting of analytic $X(\mu)$ in (4-9) and $\mu=\gamma^{-2}$, $\tilde{X}(\gamma)$ is analytic on the interval with nonzero γ. With the chain rule of differentiation, we have

$$\frac{d\tilde{X}(\gamma)}{d\gamma} = \frac{dX(\mu)}{d\mu}\frac{d\mu}{d\gamma} = -2\gamma^{-3}\frac{dX(\mu)}{d\mu}$$

and

$$\frac{d^2\tilde{X}(\gamma)}{d\gamma^2} = \frac{d^2X(\mu)}{d\mu^2}\left(\frac{d\mu}{d\gamma}\right)^2 + \frac{dX(\mu)}{d\mu}\frac{d^2\mu}{d\gamma^2}$$

which imply (4-28) and (4-29) respectively. Q.E.D.

Based on Corollary 4.1, we have the following theorem.

Theorem 4.3: On $(\alpha_x, +\infty)$, $\tilde{X}(\gamma)$ is invertible almost everywhere.

Proof:

Since $\tilde{X}(\gamma)$ is an analytic function, it is well-known that the eigenvalues and eigenvectors of $\tilde{X}(\gamma)$ are analytic functions of γ. Consider

$$(I\lambda - \tilde{X}(\gamma))\, v = 0$$

where λ is an eigenvalue of $\tilde{X}(\gamma)$ and v is a corresponding eigenvector. Then it is easy to have

$$(I\dot{\lambda} - \dot{\tilde{X}})\, v + (I\lambda - \tilde{X})\, \dot{v} = 0 \tag{4-30}$$

which implies

$$\dot{\lambda}(\gamma) = \frac{v^T \dot{\tilde{X}}\, v}{v^T v} \geq 0, \tag{4-31}$$

Let $\lambda_i(\gamma)$ be the ith eigenvalue of $\tilde{X}(\gamma)$, i = 1, 2, ..., n, where n is the dimension of $\tilde{X}(\gamma)$. If there is a $\gamma_1 \in (\alpha_x, +\infty)$ such that $\lambda_i(\gamma_1) = 0$, then define a new function on $(\alpha_x, +\infty)$ as following

$$f(\gamma) := v_i(\gamma_1)^T\, \tilde{X}(\gamma)\, v_i(\gamma_1) \tag{4-32}$$

where $v_i(\gamma_1)$ is an eigenvector corresponding to $\lambda_i(\gamma_1)$. It is trivial to show that $f(\gamma)$ is a nondecreasing concave function on $(\alpha_x, +\infty)$. Since $f(+\infty) > 0$ because $\tilde{X}(+\infty) > 0$ under the assumption we made at the beginning of this section, we can conclude that

$$\dot{f}(\gamma_1) = \frac{df(\gamma_1)}{d\gamma} > 0. \tag{4-33}$$

Based on eq. (4-31) and eq. (4-32), eq. (4-33) implies

$$\dot{\lambda}_i(\gamma_1) > 0. \tag{4-34}$$

By the property that $\dot{\lambda}_i(\gamma_1) > 0$, with $\gamma_1 \in (\alpha_x, +\infty)$ satisfying $\lambda_i(\gamma_1) = 0$, it comes the conclusion that $\tilde{X}(\gamma)$ has at most n singularity points on $(\alpha_x, +\infty)$. In other words, $\tilde{X}(\gamma)$ is invertible almost everywhere on $(\alpha_x, +\infty)$. Q.E.D.

In the same way, define

$$\alpha_y := \inf \{\gamma : \gamma \in \mathbb{R}_+, \gamma > \overline{\sigma}(\tilde{D}_\perp D_{11}{}^T) \text{ and } Y_\infty \text{ exists} \}$$

and let Y be an antistabilizing solution to the Riccati equation corresponding to the following Hamiltonian

$$-\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} J(\mu) \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}.$$

Then we have the following theorem.

Theorem 4.4: On $(0, \alpha_y{}^{-2})$, $Y(\mu)$ is an analytic function which satisfies

$$\frac{dY}{d\mu} \le 0 \tag{4-35}$$

and

$$\frac{d^2Y}{d\mu^2} \le 0 \tag{4-36}$$

Define

$$\tilde{Y}(\gamma) := Y(\mu)\Big|_{\mu=\gamma^{-2}}$$

then we have following corollary.

Corollary 4.2: $\tilde{Y}(\gamma)$ is a well-defined function on $(\alpha_y, +\infty)$. Moreover, $\tilde{Y}(\gamma)$ is analytic and satisfies

$$\frac{d\tilde{Y}(\gamma)}{d\gamma} \ge 0 \tag{4-37}$$

and

$$\frac{d^2\tilde{Y}(\gamma)}{d\gamma^2} \leq 0 \tag{4-38}$$

on $(\alpha_y, +\infty)$.

Based on the above corollary, we have the following.

Theorem 4.5: On $(\alpha_y, +\infty)$, $\tilde{Y}(\gamma)$ is invertible almost everywhere.

Now, we are ready to show that $X_\infty(\gamma)$ and $Y_\infty(\gamma)$ are nonincreasing convex functions in the domain of interest. Without loss of generality, the assumptions that $(D_\perp^T C_1, -A+B_2D_{12}^TC_1)$ is detectable and $(-A+B_1D_{21}^TC_2, B_1\tilde{D}_\perp^T)$ is stabilizable are still used in the following to simplify the proof.

From eq. (4-9), it is clear that if $\tilde{X}(\gamma) = X(\mu)$ is invertible then $X_\infty(\gamma) = \tilde{X}(\gamma)^{-1}$. Although, $\tilde{X}(\gamma)$ is not always invertible on $(\alpha_x, +\infty)$, from Theorem 4.3, it can only lose rank at some specific isolated points. Since $\tilde{X}(\gamma)$ is a nondecreasing concave function as mentioned in Corollary 4.1, there must exist β_x such that $\tilde{X}(\gamma)$ is positive definite when $\gamma > \beta_x$. That is,

$$\beta_x := \inf\{\gamma : \gamma \in \mathbb{R}_+, \gamma \geq \alpha_x \text{ and } \tilde{X}(\gamma) \text{ is positive definite}\}.$$

This means, on the interval $(\beta_x, +\infty)$, $\tilde{X}(\gamma)$ is positive definite.

Theorem 4.6: On $(\beta_x, +\infty)$, X_∞ is an analytic nonincreasing convex function of γ, i.e.,

$$\dot{X}_\infty = \frac{dX_\infty}{d\gamma} \leq 0 \tag{4-39}$$

and

$$\ddot{X}_\infty = \frac{d^2X_\infty}{d\gamma^2} \geq 0 \tag{4-40}$$

Proof:

Because $\tilde{X}(\gamma)$ is positive definite and $X_\infty = \tilde{X}(\gamma)^{-1}$, by the chain rule of differentiation and Corollary 4.1, we have

$$\frac{dX_\infty}{d\gamma} = \frac{d(\tilde{X}^{-1}(\gamma))}{d\gamma} = -X_\infty\frac{d\tilde{X}}{d\gamma}X_\infty \leq 0 \tag{4-41}$$

and

$$\frac{d^2X_\infty}{d\gamma^2}=\frac{d(\frac{dX_\infty}{d\gamma})}{d\gamma}=-\frac{d(X_\infty\frac{d\tilde{X}}{d\gamma}X_\infty)}{d\gamma}=X_\infty(2\frac{d\tilde{X}}{d\gamma}X_\infty\frac{d\tilde{X}}{d\gamma}-\frac{d^2\tilde{X}}{d\gamma^2})X_\infty \tag{4-42}$$

Since X_∞ is positive definite and $\frac{d^2\tilde{X}}{d\gamma^2}$ is negative semi-definitive, it turns out that

$$\ddot{X}_\infty=\frac{d^2X_\infty}{d\gamma^2}\geq 0 \qquad \text{Q.E.D.}$$

Similarly, by defining

$\beta_y := \inf\{\gamma : \gamma \in \mathbf{R}_+, \gamma \geq \alpha_y \text{ and } \tilde{Y}(\gamma) \text{ is positive definite}\}$

we have the following theorem for Y_∞.

Theorem 4.7: On $(\beta_y, +\infty)$, Y_∞ is an analytic nonincreasing convex function of γ, i.e.,

$$\dot{Y}_\infty=\frac{dY_\infty}{d\gamma}\leq 0 \tag{4-43}$$

and

$$\ddot{Y}_\infty=\frac{d^2Y_\infty}{d\gamma^2}\geq 0 \tag{4-44}$$

At the beginning of this section, we assumed that $(D_\perp^T C_1, -A+BR^{-1}D_{1\bullet}^T C_1)$ is detectable, or equivalently that $(D_\perp^T C_1, -A+B_2D_{12}^TC_1)$ is detectable. This assumption can be removed by the following arrangement. If $(D_\perp^T C_1, -\hat{A})$ is not detectable, where

$\hat{A} = A+BD_{12}^T C_1,$

one can always find an orthogonal matrix $U = [U_1\ U_2]$ such that

$$U^T\hat{A}U=\begin{bmatrix} U_1^T\hat{A}U_1 & 0 \\ U_2^T\hat{A}U_1 & U_2^T\hat{A}U_2 \end{bmatrix},$$

$$D_\perp^T C_1U = [D_\perp^T C_1U_1 \quad 0],$$

with $(D_\perp^TC_1U_1, -U_1^T\hat{A}U_1)$ detectable and $X_\infty = \mathrm{Ric}(H_\infty)$ can be expressed as

$$X_\infty = U\begin{bmatrix} X_1 & 0 \\ 0 & 0 \end{bmatrix}U^T.$$

Note that U is not a function of γ, and X_1 is the Riccati solution of eq. (2-5a) with (A, B, C_1) replaced by $(U_1^T\hat{A}U_1, U_1^TB, C_1U_1)$. Therefore, no matter whether $(D_\perp^T C_1, -\hat{A})$ is detectable or not, it is always true that $X_\infty(\gamma)$ exists almost everywhere on $(\alpha_x, +\infty)$ or X_1 is nonsingular almost everywhere on $(\alpha_x, +\infty)$. Furthermore, X_1 is positive definite on $(\beta_x, +\infty)$.

Define

$$\alpha := \max\{\alpha_x, \alpha_y\},$$

and

$$\beta := \max\{\beta_x, \beta_y\},$$

then X_∞ and Y_∞ exist on $(\alpha, +\infty)$ and are positive semi-definite $(\beta, +\infty)$.

The following theorem, which is one of the main results in Li and Chang [16], showed that if X_∞ and Y_∞ are nonincreasing convex functions of γ, then so is $\rho[X_\infty Y_\infty]$.

Theorem 4.8: Let β be the infimum of γ such that both X_∞ and Y_∞ exist and are positive semi-definite. On $(\beta, +\infty)$,
a) all eigenvalues of $X_\infty Y_\infty$ are smooth, nonincreasing functions of γ;
b) $\rho(X_\infty Y_\infty)$ is a nonincreasing, convex function of γ.

Proof:

a) It is quite straightforward to show that for any nontrivial eigenvalue of $X_\infty Y_\infty$, we have

$$\lambda[X_\infty(\gamma)Y_\infty(\gamma)] = \lambda[V^TX_\infty(\gamma)Y_\infty(\gamma)V] = \lambda(V^TU\begin{bmatrix} X_1(\gamma) & 0 \\ 0 & 0 \end{bmatrix}U^TV\begin{bmatrix} Y_1(\gamma) & 0 \\ 0 & 0 \end{bmatrix})$$
$$= \lambda(\begin{bmatrix} V_1^TU_1X_1(\gamma)U_1^TV_1Y_1(\gamma) & 0 \\ 0 & 0 \end{bmatrix}) = \lambda[Z(\gamma)Y_1(\gamma)],$$

where $Z(\gamma) = Z^T(\gamma) = V_1^TU_1X_1(\gamma)U_1^TV_1$ has following properties

$$Z(\gamma) \geq 0, \tag{4-45a}$$

$$\mathring{Z}(\gamma) \leq 0 \tag{4-45b}$$

and

$$\ddot{Z}(\gamma) \geq 0 \tag{4-45c}$$

on $(\beta, +\infty)$. Now consider

$$[I\lambda - Z(\gamma)Y_1(\gamma)]\, w = 0, \tag{4-46}$$

where λ is any eigenvalue of $Z(\gamma)Y_1(\gamma)$ and w a corresponding eigenvector. Note that this equation is equivalent to

$$[\lambda Y_1^{-1}(\gamma) - Z(\gamma)]\, u = 0, \tag{4-47a}$$

or

$$[\lambda \hat{Y}(\gamma) - Z(\gamma)]\, u = 0 \tag{4-47b}$$

with $\hat{Y}(\gamma) = Y_1^{-1}(\gamma)$ and $u = Y_1 w$. From (4-47), we have

$$(\dot{\lambda}\hat{Y} + \lambda\dot{\hat{Y}} - \dot{Z})\, u + (\lambda\hat{Y} - Z)\, \dot{u} = 0 \tag{4-48}$$

and

$$u^T(\dot{\lambda}\hat{Y} + \lambda\dot{\hat{Y}} - \dot{Z})\, u = 0. \tag{4-49}$$

Then it is easy to show from these two equations that

$$\dot{\lambda}(\gamma) = \frac{u^T(\dot{Z} - \lambda\dot{\hat{Y}})u}{u^T\hat{Y}u} \leq 0 \tag{4-50}$$

on $(\beta, +\infty)$, since $\dot{Z} \leq 0$, $\hat{Y} \geq 0$ and $\dot{\hat{Y}} \geq 0$ (see (4-45b) and (4-37) in Corollary 4.2).

b) Next is the proof of the convexity of $\rho(X_{-\infty}Y_{-\infty})$. From the equations (4-48) and (4-49) with λ replaced by the maximum eigenvalue of $X_{-\infty}Y_{-\infty}$, we come to the following two equations:

$$\dot{u}^T(\dot{\lambda}_{max}\hat{Y} + \lambda_{max}\dot{\hat{Y}} - \dot{Z})\, u = -\dot{u}^T(\lambda_{max}\hat{Y} - Z)\, \dot{u} \tag{4-51}$$

$$\dot{u}^T(\dot{\lambda}_{max}\hat{Y} + \lambda_{max}\dot{\hat{Y}} - \dot{Z})\, u + u^T(\dot{\lambda}_{max}\hat{Y} + \lambda_{max}\dot{\hat{Y}} - \dot{Z})\, \dot{u}$$
$$+ u^T(\ddot{\lambda}_{max}\hat{Y} + 2\dot{\lambda}_{max}\dot{\hat{Y}} + \lambda_{max}\ddot{\hat{Y}} - \ddot{Z})\, u = 0. \tag{4-52}$$

Combining the above two equations, we have

$$\ddot{\lambda}_{max}(\gamma) = \frac{2\dot{u}^T(\lambda_{max}\hat{Y} - Z)\dot{u} + u^T(\ddot{Z} - 2\dot{\lambda}_{max}\dot{\hat{Y}} - \lambda_{max}\ddot{\hat{Y}})u}{u^T\hat{Y}u}. \tag{4-53}$$

Note that $(\lambda_{max}\hat{Y}\text{-}Z) \geq 0$, $\ddot{Z} \geq 0$, $\dot{\hat{Y}} \geq 0$, $\dot{\lambda}_{max}(\gamma) \leq 0$ and $\ddot{\hat{Y}} \leq 0$, we have $\ddot{\lambda}_{max}(\gamma) \geq 0$ and therefore $\ddot{\rho}\,[X_\infty(\gamma)Y_\infty(\gamma)] \geq 0$ on $(\beta, +\infty)$. Hence we complete the proof that $\rho(X_\infty Y_\infty)$ is a convex function of γ on $(\beta, +\infty)$. Q.E.D.

The properties of the H^∞ Riccati solutions presented in this section will be employed in Chapter 5 to develop efficient algorithms for computing the optimal H^∞ norm.

CHAPTER 5

COMPUTATION OF THE OPTIMAL H^∞ NORM

Recently, an efficient algorithm for computing the optimal H^∞ norm was proposed by Scherer [15]. Scherer considered the inverse (or pseudo inverse) of the DGKF H^∞ Riccati solutions, $X_\infty(\gamma)$ and $Y_\infty(\gamma)$, defined a new independent variable $\mu = \gamma^{-2}$, and showed that these inverses are concave functions of μ in the matrix sense on their domains of definition. Based on this fact, a quadratically convergent Newton-like algorithm was proposed to compute the optimal H^∞ norm.

Pandey et. al.'s hybrid gradient-bisection method [13] and Chang et. al.'s double secant and bisection method [14] were also proposed for the computation of the optimal H^∞ norm. The significance of the conjecture that $\rho(\gamma) := \rho[X_\infty(\gamma)Y_\infty(\gamma)]$, the spectral radius of $X_\infty(\gamma)Y_\infty(\gamma)$, is a convex function of γ^2 was mentioned in these two papers. Since there was no proof for this conjecture, bisection was used in these two algorithms as supplement to guarantee convergence.

Based on the properties of the Riccati solutions described in the previous chapter, a quadratically convergent algorithm can be easily developed to compute the optimal H^∞ norm. According to Theorem 2.1, we can see that finding the optimal H_∞ norm, denoted as γ_∞, is equivalent to finding the infimum γ such that all four conditions in (2-6) hold. From the previous chapter, it is obvious that $\gamma_\infty \in [\beta, +\infty)$. It is possible for β to be γ_∞, especially when β and α are identical, however, with very few exceptions, $\gamma_\infty \in (\beta, +\infty)$, which implies that γ_∞ is the solution to $\rho(\gamma) = \gamma^2$. The relations between α, β and γ_∞ are shown in the figure below.

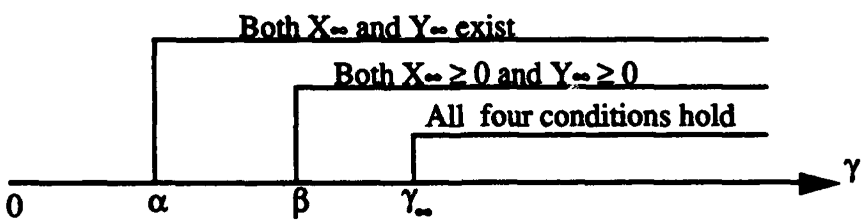


Fig. 5.1

5.1 Computation of γ_∞

Fig. 5.1 implies that the problem of finding the optimal γ_∞ is actually that of either

searching for the intersection point of $\rho(\gamma)$ with γ^2 inside $(\beta, +\infty)$ or computing the boundary point β. The monotonicity and convexity of $\rho(\gamma)$ in Theorem 4.8 suggests that if we use a gradient method to search for γ_∞ with a starting point γ_n inside the interval (β, γ_∞), then the convergence is guaranteed.

The derivative of $\rho(\gamma)$ at γ_n can be computed as follows,

$$\dot{\rho}(\gamma) = \frac{v^T(\dot{X}_\infty Y_\infty + X_\infty \dot{Y}_\infty)w}{v^T w} \tag{5.1}$$

where v and w are left and right eigenvectors of $X_\infty Y_\infty$ respectively corresponding to its maximal eigenvalue. $\dot{X}_\infty$ and $\dot{Y}_\infty$ can be obtained by solving the following Lyapunov equations:

$$A_x^T \dot{X}_\infty + \dot{X}_\infty A_x + (X_\infty B + C_1^T D_{1\cdot}) R^{-1} \dot{R} R^{-1} (X_\infty B + C_1^T D_{1\cdot})^T = 0 \tag{5.2a}$$

$$A_y \dot{Y}_\infty + \dot{Y}_\infty A_y^T + (CY_\infty + D_{\cdot 1} B_1^T)^T \tilde{R}^{-1} \dot{\tilde{R}} \tilde{R}^{-1} (CY_\infty + D_{\cdot 1} B_1) = 0 \tag{5.2b}$$

with

$$A_x = A - BR^{-1}D_{1\cdot}^T C_1 - BR^{-1}B^T X_\infty$$

and

$$A_y = A - B_1 D_{\cdot 1}^T \tilde{R}^{-1} C - C^T \tilde{R}^{-1} C Y_\infty$$

$$\dot{R} = \begin{bmatrix} -2\gamma I_{m1} & 0 \\ 0 & 0 \end{bmatrix} \text{ and } \dot{\tilde{R}} = \begin{bmatrix} -2\gamma I_{p1} & 0 \\ 0 & 0 \end{bmatrix};$$

Assume that we have a starting point γ_n in the interval (β, γ_∞), then the optimal γ can be obtained easily as follows. Refer to Fig. 5.2, draw the tangent line with slope $\dot{\rho}(\gamma_n)$, which can be computed by equations (5-1), passing through the point $(\gamma_n, \rho(\gamma_n))$. The abscissa, γ_{n+1}, of the intersection of the tangent line and the curve $y = \gamma^2$, always lies between γ_n and γ_∞. The search process is repeated until the gap $\gamma_\infty - \gamma_{n+1}$ is small enough.

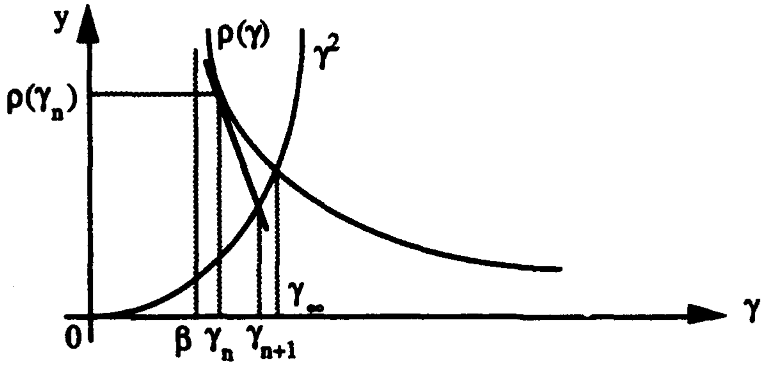


Fig. 5.2

Furthermore, we will see that the convergence rate is quadratic. Define $\varepsilon_n = \gamma_\infty - \gamma_n$ and $\varepsilon_{n+1} = \gamma_\infty - \gamma_{n+1}$. It is straightforward to show that

$$\varepsilon_{n+1} \approx \frac{\ddot{\rho}(\gamma_\infty)}{2 \mid \dot{\rho}(\gamma_\infty) - 2\gamma_\infty \mid} \varepsilon_n^2 \tag{5-3}$$

which implies quadratic convergence.

In the above algorithm of computing γ_∞, we assumed that a γ_n inside the interval (β, γ_∞) is available to start with. To find a γ_n inside this interval, usually we arbitrarily pick up a relatively large γ_1 at which the two Riccati equations (2-5a and b) have positive semidefinite solutions $X_\infty(\gamma_1)$ and $Y_\infty(\gamma_1)$. Refer to Fig. 5.3, draw a line passing through the point (γ_1, $\rho(\gamma_1)$) with slope $\dot{\rho}(\gamma_1)$. The abscissa, γ_2, of the intersection of the straight line and the curve $y = \gamma^2$, is always less than γ_∞. If $\gamma_2 > \beta$, i.e., the two Riccati solutions $X_\infty(\gamma_2)$ and $Y_\infty(\gamma_2)$ exist and are positive semidefinite, then we are ready to use the above quadratically convergent algorithm to compute γ_∞.

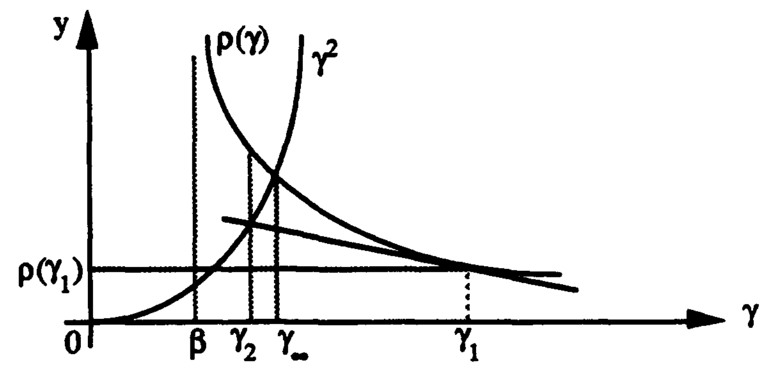


Fig. 5.3

Most of the time, the method described in the previous paragraph gives a γ_n inside the interval (β, γ_{∞}) without the computation of β. However, this method may fail. If a starting γ_1 gives a $\gamma_2 < \beta$, i.e., either $X_{\infty}(\gamma_2)$ or $Y_{\infty}(\gamma_2)$ does not exist or is not positive semidefinite, one may suggest that a smaller γ_1 could produce a $\gamma_2 > \beta$. For the case described by Fig. 5.4, a smaller γ_1 could do the job. However, for the case of Fig. 5.5, γ_2 is always less than α, and therefore less than β, no matter how small γ_1 is. Since it is difficult to tell which case we are facing and there is no efficient guideline to reduce γ_1, we suggest to compute β (or α and β) if one or two trials of γ_1 does not give a $\gamma_2 > \beta$. If either $X_{\infty}(\gamma_2)$ or $Y_{\infty}(\gamma_2)$ does not exist, we will compute α first, then compute β if necessary. The computation of α will be given later in Section 5.3. If both $X_{\infty}(\gamma_2)$ and $Y_{\infty}(\gamma_2)$ exist but are not all positive semidefinite, then γ_2 is inside the interval (α, β). A quadratically convergent algorithm for the computation of β is given in Section 5.2. Once β is obtained, then $\beta+\varepsilon$ can be served as the starting point γ_n in Fig. 5.2, where ε is a very small positive real number.

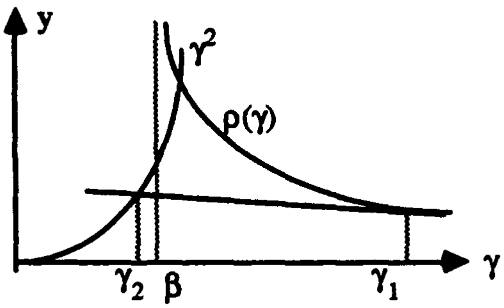


Fig. 5.4

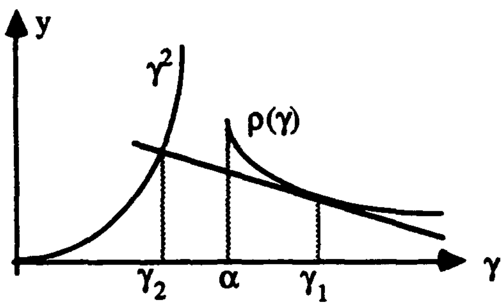


Fig. 5.5

5.2 Computation of β

In this section, we have the assumption of having a starting point inside the interval (α_x, β_x) (resp. (α_y, β_y)), which implies $\alpha_x \neq \beta_x$ (resp. $\alpha_y \neq \beta_y$). We will show how to compute α in the next section. Recall that β is equal to $\max\{\beta_x, \beta_y\}$ where β_x and β_y are infimums of γ such that (2-6b) and (2-6c) are true respectively. Since the computation of β_y is similar to that of β_x, only the algorithm for β_x is given in the following.

Without loss of generality, we assume that $(D_{\perp}{}^T C_1, - A+B_2D_{12}{}^TC_1)$ is detectable and $\tilde{X}(\gamma) := X(\mu)\big|_{\mu=\gamma^{-2}}$ where $X(\mu)$ is the antistabilizing solution of (4-9). Define $e_x(\gamma) := \lambda_{min}(\tilde{X})$ and $\dot{e}_x := \dfrac{de_x(\gamma)}{d\gamma}$ on $(\alpha_x, +\infty)$, according to the proof of Theorem 4.3 in the

previous chapter, $e_x(\gamma)$ is a nondecreasing concave function of γ on $(\alpha_x, +\infty)$ and β_x is the γ such that $e_x(\gamma)$ equals to zero. Then Newton's search scheme can be used to compute β_x.

With a starting point β_x^0 which is slightly greater than α_x, we can compute β_x by the following iteration,

$$\beta_x^{n+1} = \beta_x^n - \frac{e_x(\gamma_n)}{\dot{e}_x(\gamma_n)}$$

where

$$\dot{e}_x(\gamma) = \frac{w^T \frac{d\tilde{X}}{d\gamma} w}{w^T w}$$

In the above, w is an eigenvector of $\tilde{X}$ corresponding to its minimal eigenvalue $\lambda_{min}(\tilde{X})$ and $\frac{d\tilde{X}}{d\gamma}$ can be determined by

$$\frac{d\tilde{X}(\gamma)}{d\gamma} = \frac{dX(\mu)}{d\mu}\frac{d\mu}{d\gamma} = -2\gamma^{-3}\frac{dX(\mu)}{d\mu}$$

and

$$\frac{dX(\mu)}{d\mu} = \sum_{i=1}^{m1} \dot{X}_{\mu i} \frac{1}{(1-\lambda_i\mu)^2} \leq 0$$

where $\dot{X}_{\mu i}$ can be obtain from (4-16).

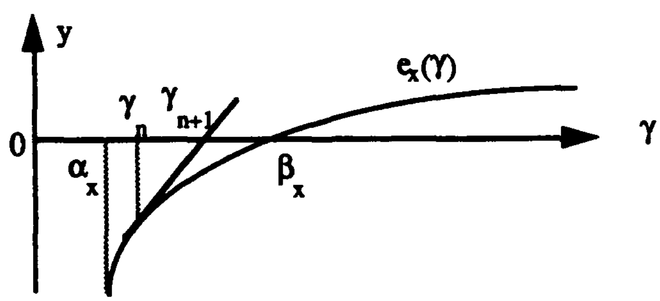


Fig. 5.6 Newton algorithm for searching β_x.

As we assumed, we have a γ_n inside the interval (α_x, β_x), i.e., $\tilde{X}(\gamma_n)$ exists and $e_x(\gamma_n) < 0$. Refer to Fig. 5.6, draw a line with slope $\dot{e}_x(\gamma_n)$ passing through the point $(\gamma_n, e_x(\gamma_n))$. This straight line will intersect the horizontal line y = 0 at γ_{n+1} which always lies

between γ_n and β_x. The search process is repeated until the gap β_x-γ_{n+1} is small enough. Hence, the convergence is guaranteed. Furthermore, we will see that the convergence rate is quadratic. Define $\varepsilon_n = \beta_x - \gamma_n$ and $\varepsilon_{n+1} = \beta_x - \gamma_{n+1}$. It is straightforward to show that

$$\varepsilon_{n+1} \approx \frac{|\ \ddot{e}_x(\beta_x)\ |}{|\ 2\ \dot{e}_x(\beta_x)\ |}\ \varepsilon_n^2 \tag{5-4}$$

which implies quadratic convergence.

In the above algorithm of computing β_x, we assumed that a γ_n inside the interval (α_x, β_x) is available to start with. This γ_n could be obtained in Section 5.1, i.e. γ_2 in Fig 5.3. However, if either $X_\infty(\gamma_2)$ or $Y_\infty(\gamma_2)$ does not exist, i.e., if $\gamma_2 < \alpha$, we will compute α first.

5.3 Computation of α

Recall that α_x (resp. α_y) is the infimum of γ such that the Hamiltonian matrix $H_\infty(\gamma)$ in (2-5a) (resp. $J_\infty(\gamma)$ in (2-5b)) has no $j\omega$-axis eigenvalues and $\alpha := \max\{\alpha_x, \alpha_y\}$. As we mentioned before, it is possible for α to be the optimal γ, i.e., all the four conditions in (2-6) hold at this point. If this is not the case, α can be used as a starting point to search for β in Section 5.2 and $\beta+\varepsilon$ in turn can serve as a starting point to search for γ_∞ in Section 5.1, if β is not the optimal γ. In the following, only the computation of α_x will be discussed, since the computation of α_y is similar to that of α_x.

One can easily have a *bisection iterative algorithm* to search for the infimum of γ such that the Hamiltonian matrix $H_\infty(\gamma)$ has no $j\omega$-axis eigenvalues. This method is slow and therefore not recommended.

To have a more efficient algorithm, we need the fact that α_x can be also expressed as the supremum of a frequency function. According to [14],

$$\alpha_x^2 = \sup_{\omega} \{ \lambda_{max} G_1^*[I - G_2(G_2^* G_2)^{-1} G_2^*] G_1(j\omega) \}$$

where $*$ is conjugate transpose and $G_1(s)$ and $G_2(s)$ are given by

$$\begin{bmatrix} G_1(s) & G_2(s) \end{bmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \end{array}\right]$$

By defining

$$h(\omega) := \lambda_{max} G_1^*[I - G_2(G_2^*G_2)^{-1}G_2^*]G_1\,(j\omega)$$

the problem of finding α_x becomes that of finding the supremum of the function $h(\omega)$.

There are several efficient algorithms available for searching for the supremum of $h(\omega)$ [24,25]. We can start from arbitrarily choosing a frequency, say ω_1. Let $\gamma = h(\omega_1)$ and then find all the positive real ω's such that $h(\omega) = \gamma$. These ω's can be easily obtained from computing the $j\omega$-axis eigenvalues of the Hamiltonian matrix $H_\infty(\gamma)$. Now, we have the frequency intervals in which $h(\omega) \geq \gamma$. Evaluate $h(\omega)$ for each midpoint of these frequency intervals and update γ to be the maximum of these $h(\omega)$'s. Then, find the new frequency intervals in which $h(\omega) \geq \gamma$. According to [25], the convergence of this iterative process is quadratic. This process can be repeated until only one frequency interval with $h(\omega) \geq \gamma$ is left and the interval length is negligible [25]. In [24], this process is terminated when a frequency interval in which $h(\omega)$ is convex and greater than γ is found. A search method called Brent method was used to search for the supremum of $h(\omega)$ in the convex frequency interval.

5.4 An Illustrative Example

The following is a simple H^∞ optimization problem which is used to illustrate the proposed algorithm of computing the optimal H^∞ norm. A realization of the generalized plant G(s) is given by

$$G(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] = \left[\begin{array}{cc|cc|c} -1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 \\ \hline 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \hline 1 & 1 & 0 & 1 & 0 \end{array}\right].$$

Starting from $\gamma_0 = 100$, we have $\rho(\gamma_0) = \rho[X_\infty(\gamma_0)Y_\infty(\gamma_0)] = 2.16e01$. The slope of $\rho(\gamma)$ at this point $(\gamma_0, \rho(\gamma_0))$ is $\mathring{\rho}\,(\gamma_0) = -3.49e\text{-}05$. The tangent line at this point will intersect with the curve $y = \gamma^2$ at $\gamma_1 = 4.65$. Again, evaluate $\rho(\gamma_1) = 2.24e01$ and compute the slope $\mathring{\rho}\,(\gamma)$ at $(\gamma_1, \rho(\gamma_1))$. Since $\rho(\gamma_1) > \gamma_1^2$, γ_1 is inside the interval (β, γ_∞) and therefore from now on the convergence is guaranteed. The process is repeated until the gap between $\sqrt{\rho(\gamma_n)}$ and γ_n is small enough. The following data show that only four iterations are needed to reach the optimum, $\gamma_{op} = 4.734160476390407$, with accuracy better than 10^{-14}.

Iter	γ_n	$\lvert\sqrt{\rho(\gamma_n)} - \gamma_n\rvert$	$\dot{\rho}(\gamma_n)$
0	100	9.535e+01	-3.487e-05
1	4.647998761538930e+00	8.403e-01	-3.811e-01
2	4.734064423866624e+00	9.440e-04	-3.595e-01
3	4.734160476276923e+00	1.115e-09	-3.595e-01
4	4.734160476390407e+00	7.105e-15	

By the formulas in (2-8) and the descriptor-form technique described in Section 2.2, we are able to construct an optimal controller as follows:

$$K_{opt}(s) = \left[\begin{array}{c|c} -.87542 & -0.13925 \\ \hline 4.42042 & -4.73416 \end{array}\right]$$

with the H^∞ norm of the closed-loop system equals γ_{op}. Note that the optimal controller has a direct feedthrough term and thus has infinite bandwidth. If we choose $\gamma = 4.8$ which is about 1.4% higher than γ_{op}, we have a suboptimal controller

$$K_{sub}(s) = \left[\begin{array}{cc|c} -8.67072e-01 & 1.32928e-01 & -1.38959e-01 \\ -1.38320e+01 & -1.52323e+02 & 4.73733e+00 \\ \hline -9.30025e+00 & -1.49792e+02 & 0 \end{array}\right]$$

which has a reasonable bandwidth and the closed-loop H^∞ norm, $\|T_{zv}\|_\infty < 4.8$ which is only 1.4% away from the optimal H^∞ norm.

CHAPTER 6

AN APPLICATION OF μ-SYNTHESIS TO A ROBUST FLIGHT CONTROL PROBLEM

6.1 Introduction

In general, a system with uncertainties can be divided into two parts, the nominal plant M(s) and uncertainties Δ. If Δ is norm bounded but otherwise unconstrained, it is called unstructured uncertainty. However, if Δ has a structure constraint, e.g., Δ is diagonal, then it is called structured uncertainty. Obviously, the set of unstructured uncertainties is larger than that of structured uncertainties. H^∞ design technique [1,2,7,8] can be used to solve robust stability problems with unstructured uncertainties, whereas μ-synthesis technique is used to handle systems with structured uncertainties. Since the structure information of the uncertainty is used in μ-synthesis, a less conservative solution can be obtained. Moreover, robust performance problem can also be formulated as a structured uncertainty problem and solved by μ-synthesis.

To use μ-synthesis, one needs to formulate the problem first and put it into a Linear Fractional Transformation (LFT) [2] form, while choosing weighting matrices for certain design specifications. With this augmented plant, D-K iteration design algorithm is employed to solve the μ-synthesis problem. The D-K iteration consists of H^∞ optimization step (K-step) and μ-analysis step (D-step). While each step is a convex optimization problem [26], the overall optimization problem is not convex. Hence a local minimal point it converges can not be guaranteed to be the global minimal point. Another comment on D-K iteration algorithm is that to have a better accuracy, a higher order curve fitting in D-step is required, which in turn gives a higher order controller. However, the high order controller usually can be reduced to that of the generalized plant without degrading the system performance.

The chapter is organized as follows. Section 6.2 gives a review of some preliminaries. We will show the problem formulation in section 6.3. In section 6.4 the μ-synthesis is used to design a robust controller. Simulations and analysis are also included

in this section. Section 6.5 is a conclusion.

6.2 Preliminaries

In this section, we will give a quick review of some fundamental knowledges including the definition of μ, robust stability, and robust performance.

Usually the mathematical representation of a system under consideration is only an approximation to the actual system. The "actual model" $\tilde{M}$ is in a neighborhood of the nominal model M. According to [4], $\tilde{M}$ can always be separated into two parts illustrated by the following block diagram with an M-Δ structure, where M is the nominal system and Δ is a block diagonal uncertainty matrix.

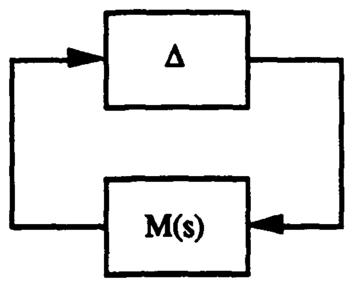


Fig. 6.1 M-Δ structure

With this block diagram, the stability analysis essentially boils down to ensuring that $I - M\Delta(j\omega)$ remains nonsingular at all frequencies under all Δ considered (small gain theorem [1]). This statement can be seen by considering the following fact. Assume M(s) is a stable system. When no perturbation is introduced, i.e, $\Delta = 0$, $\tilde{M} = M$. However, as Δ grows up, in the sense of $\bar{\sigma}(\Delta)$ (2-norm), $I - M\Delta$ may not remain nonsingular, which implies the instability of $\tilde{M}$. In the following, we will introduce the definition of μ, which measures how large Δ can be such that $\tilde{M}$ remains stable.

The structured singular value (SSV or μ) of a system M with respect to the given structure of Δ is defined by

$$\mu(M) = \left\{\min_{\Delta}\left(\bar{\sigma}(\Delta) : \det(I - M\Delta) = 0\right)\right\}^{-1} \qquad \forall\, \omega \in \mathbf{R}_+ \qquad (6\text{-}1)$$

where $\mathbf{R}_+$ is the set of positive real numbers. In the following are listed several remarks on the above definition.

- Obviously, the smaller the μ is, the larger uncertainties are allowed such that $\tilde{M}$

remains stable.

- μ depends highly on the structure of Δ; the more information used, the less conservative the solution will be. If Δ is a full block matrix, $\mu(M) = \bar{\sigma}(M)$. On the other hand, if Δ is a diagonal matrix, then $\mu(M) = \rho(M)$, where ρ denotes spectral radius [4]. Usually, Δ is in *block* diagonal form, and obviously $\rho(M) \leq \mu(M) \leq \bar{\sigma}(M)$ in general.

- The computation of μ is still an open problem. If Δ has three or less blocks, μ can be computed by

$$\mu(M) = \inf_{D}[\bar{\sigma}(DMD^{-1})], \tag{6-2}$$

where D is a positive definite block diagonal matrix with the same structure as Δ. However, for the uncertainties with more than three diagonal blocks the right hand side term of eq.(6-2) is just an upper bound.

Robust stability and robust performance

Most control problems can be represented by the following Linear Fractional Transformation (LFT) form:

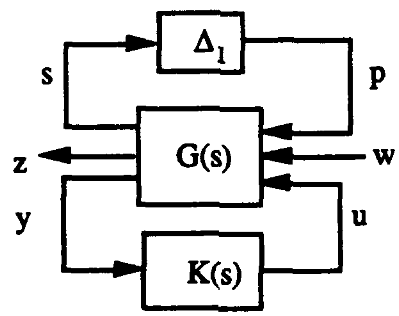


Fig. 6.2

where G(s) is the generalized plant including possible weighting matrices, z is the controlled output usually including the error signal and a weighted control input, w is the exogenous input containing the disturbances, noises and commands, and y is the measured output vector consisting of all the signals which can be measured and available for feedback, Δ_1 represents plant uncertainties. The design objective is to find a controller K(s) such that the system robust stability and robust performance can be achieved, i.e., under all the perturbations considered, the system remains internally stable and the H^∞ norm of the transfer function from w to z remains less than a prescribed value. This problem can be

solved by μ-synthesis technique on the following block diagram,

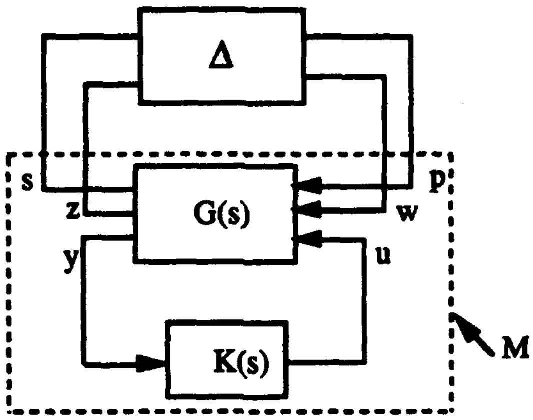


Fig.6.3

where $\Delta = \mathrm{diag}\{\Delta_1, \Delta_2\}$ and Δ_2 is a full block matrix corresponding to the transfer function of the closed loop system from w to z. If we combine G(s) and K(s) together by LFT, then it corresponds to the M shown in Fig.6.1, i.e., $M = F_l(G, K)$.

μ-synthesis and D-K iteration

The original concept of μ-synthesis is to design a controller K(s) such that μ of $M = F_l(G, K)$ is minimized at all frequencies, with respect to certain Δ structure. According to the main loop theorem [6], robust stability and robust performance can be achieved if the μ-synthesis is applied on $\Delta = \mathrm{diag}\{\Delta_1, \Delta_2\}$ of the above block diagram, where Δ_1 is uncertainty block and Δ_2 performance block. However, due to the difficulty of computing μ, the μ-synthesis is relaxed to the problem of minimizing its upper bound:

$$\inf_{K} \sup_{\omega} \inf_{D} \{ \bar{\sigma}[DF_l(G, K)D^{-1}]\} \tag{6-3}$$

where D is the scaling matrix mentioned in eq.(6-2). Then, eq.(6-3) can be solved by the so called D-K iteration algorithm, which is the only solution available to the problem up to now. Basically, D-K iteration algorithm solves

$$\min_{K(s),\, D(s)} \| D(s)F_l(G(s), K(s))D^{-1}(s) \|_\infty, \tag{6-4}$$

where both D(s) and $D^{-1}(s)$ are proper stable rational functions. This optimization is a convex searching problem, if either K(s) or D(s) is fixed. Unfortunately, it is not a convex problem over D and K.

D-K iteration algorithm

Step 1: $D(s) = I$;

Step 2: Find a controller $K(s)$ such that $\|D(s)F_l[G(s), K(s)]D^{-1}(s)\|_\infty$ is minimized;

Step 3: Find constant D's at each frequency such that $\bar{\sigma}(DF_l(P, K)D^{-1})$ is minimized;

Step 4: Curve fitting for $D(s)$, where $D(s)$ is chosen such that both $D(s)$ and $D^{-1}(s)$ are proper stable rational functions;

Step 5: Go to step 2, until the local minimum is reached.

6.3 Problem Formulation

In this section, we will design a robust controller for a fighter aircraft at an altitude of 10,000 feet and a Mach Number of 0.18. The angle of attack is approximately 70 degree, and the dynamic pressure is 33.6 lb/ft^2. The system is described by the following block diagram.

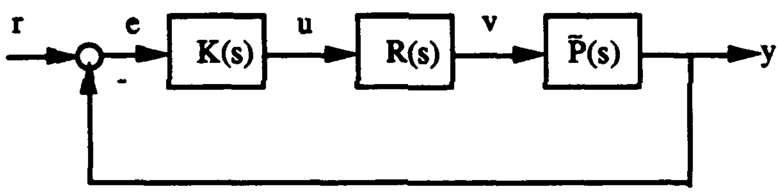


Fig.6.4

The $\tilde{P}(s)$ is longitudinal model given by state space model [9]:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} Z_a & 1 \\ M_a & M_q \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} Z_v \\ M_v \end{bmatrix} v \tag{6-5a}$$

$$y = x_1 \tag{6-5b}$$

where x_1 is the angle of attack, x_2 is the pitch rate and v is the pitch vectoring nozzle deflection. It is assumed that M_a, M_q and M_v are subjected to 25% variations from its nominal values. The nominal plant is given by:

$$A_p = \begin{bmatrix} 0.0264 & 1 \\ -0.8810 & -0.2079 \end{bmatrix} \quad B_{pv} = \begin{bmatrix} -0.0520 \\ -4.3434 \end{bmatrix} \quad C_{py} = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad D_{Py} = 0. \tag{6-6}$$

The R is the model for the actuator from pilot stick to the vectoring nozzle. Its

transfer function is given by:

$$R(s) = \frac{400}{s^2 + 24s + 400}. \qquad (6\text{-}7)$$

From above model, one can see that the plant is stable, but has a large overshoot and oscillations, long settling time and an insufficient stability margin, which can be verified by the following step response plot, where the input is the unit step function acted on pilot stick and the output is angle of attack of the aircraft.

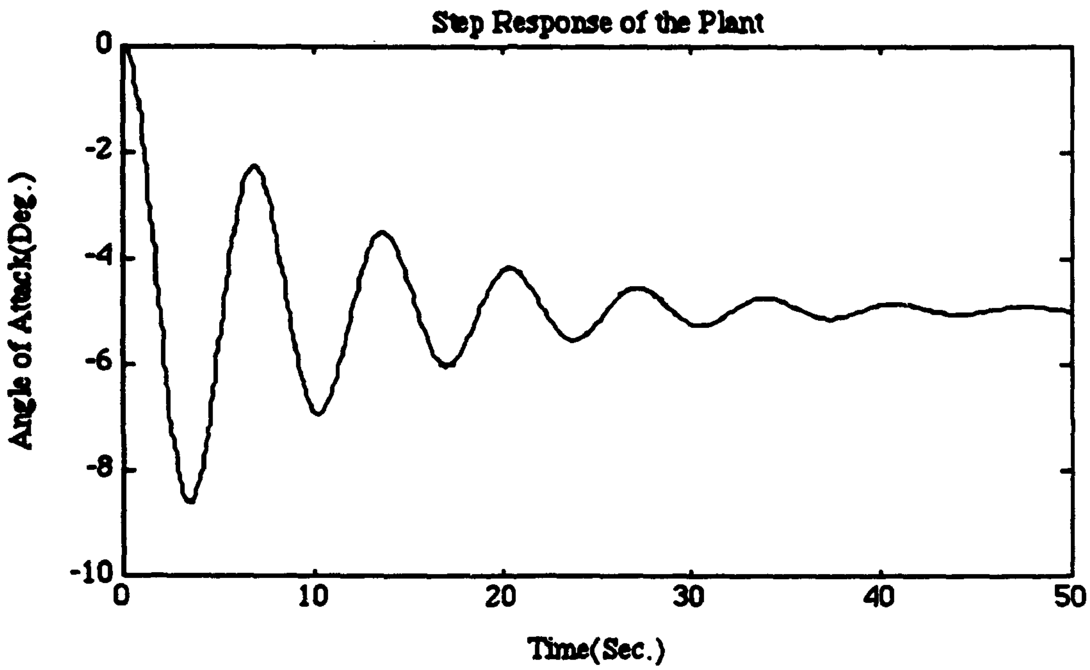


Fig. 6.5

The design specifications for the feedback system are: (1) The system has a satisfactory tracking ability with a damping ratio larger than 0.5, natural frequency large than 2.0 rad/sec, small overshoots and oscillations, and small steady-state tracking error; (2) Robustness of the system, including the stability and performance in the face of disturbance and the plant uncertainties.

In order to design a robust controller, one needs to separate plant uncertainties Δ_1 from the nominal plant. Consider eq.(6-5) and eq.(6-6), then it is easy to see that the plant can be expressed by Fig.6.6,

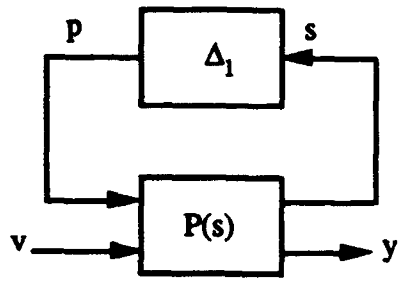


Fig. 6.6

where

$$P(s)=\left[\begin{array}{c|cc} A_p & B_{p1} & B_{p2} \\ \hline C_{p1} & D_{p11} & D_{p12} \\ C_{p2} & D_{p21} & D_{p22} \end{array}\right],\quad \Delta_1=\begin{bmatrix} \delta_1 & & \\ & \delta_2 & \\ & & \delta_3 \end{bmatrix}. \tag{6-8}$$

Here δ_1, δ_2 and δ_3 are perturbations to M_a, M_q and M_v respectively. A_P, $B_{P2} = B_{Pv}$ and $C_{P2} = C_y$ were given in eq.(6-6), and

$$B_{P1}=\begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix},\qquad C_{p1}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},\qquad D_p=\begin{bmatrix} D_{p11} & D_{p12} \\ D_{p21} & D_{p22} \end{bmatrix}=\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 \end{array}\right]. \tag{6-9}$$

With plant model and design specifications available, a controller K(s) can be designed for the following LFT block diagram, where $\overline{P}(s)$ consists of nominal plant P(s) and actuator R(s).

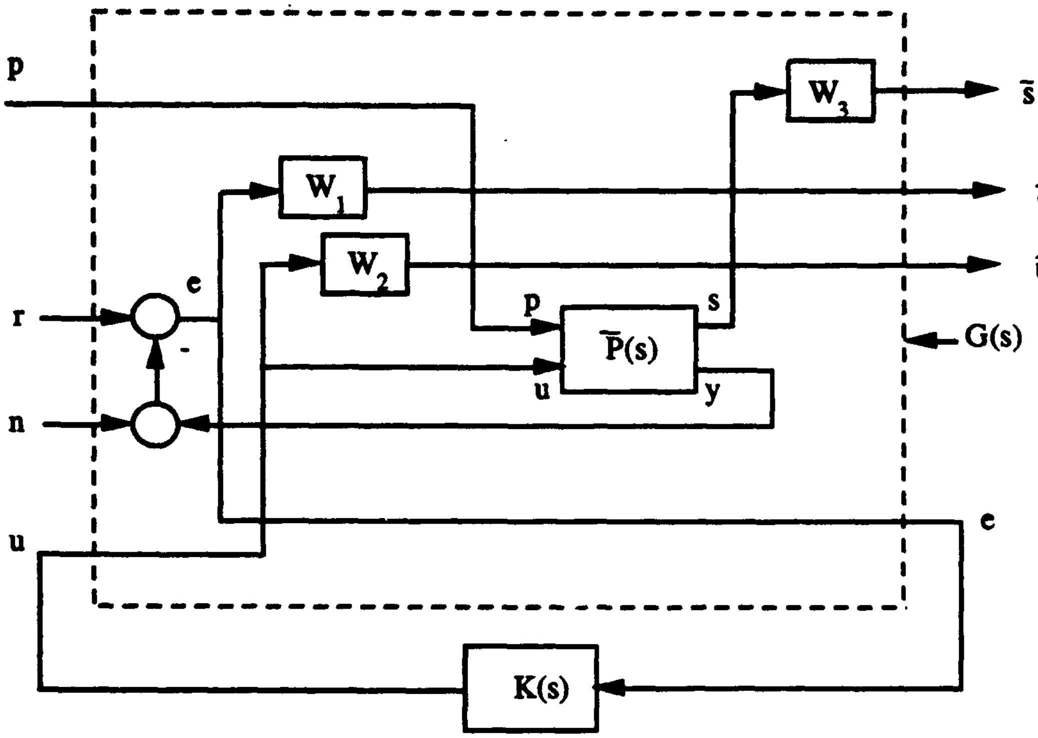


Fig.6.7

Compared with Fig.6.2, the above block diagram implies that the exogenous input w consists of command r and noise n. The controlled output z consists of weighted tracking error and weighted control constraint. The p and $\bar{s}$ are fictitious input and output reflecting system uncertainties. $W_1(s)$, $W_2(s)$ and $W_3(s)$ are the weighting matrices chosen by designers such that the design specifications can be met. $W_1(s)$ is a weighting matrix for tracking error and is chosen as a low-pass filter to emphasize the tracking accuracy at low frequencies (small steady-state error). The weighting matrix $W_2(s)$ is designed for control constraint. In the real situation, the control input must be restricted because of the limited control energy and actuator saturation. Usually $W_2(s)$ is chosen as a diagonal constant matrix; the larger $W_2(s)$ is, the less the control will be used. $W_3(s)$ is used to normalize the system uncertainties. In this particular problem, we choose

$$W_1(s) = \frac{s+10}{s+0.001},$$

$$W_2(s) = 0.5 \text{ and } W_3(s) = 0.25 \operatorname{diag}\{0.881, 0.2079, 4.3434\}.$$

With these weighting matrices, the μ-synthesis technique can be used to design a robust controller which maintains the quality of system performance and stability in the face of uncertainties and sensor noise. It is easy to derive the generalized plant G(s) from Fig 6.7:

$$G(s) = \left[\begin{array}{ccc:c} W_3\bar{P}_{11} & 0 & 0 & W_3\bar{P}_{12} \\ -W_1\bar{P}_{21} & W_1 & -W_1 & -W_1\bar{P}_{22} \\ 0 & 0 & 0 & W_2 \\ \hdashline -\bar{P}_{21} & I & -I & -\bar{P}_{22} \end{array}\right]. \tag{6-10}$$

Then by using the notations and formulas in [27], it is easy to see that the generalized plant G(s) has a state space realization as follows:

$$G(s) = \left[\begin{array}{ccc|ccc:c} A_1 & B_1C_{p2} & B_1D_{p22}C_R & B_1D_{p21} & -B_1 & B_1 & B_1D_{p22}D_R \\ 0 & A_p & B_{p2}C_R & B_{p1} & 0 & 0 & B_{p2}D_R \\ 0 & 0 & A_R & 0 & 0 & 0 & B_R \\ \hline 0 & D_3C_{p1} & D_3D_{p12}C_R & D_3D_{p11} & 0 & 0 & D_3D_{p12}D_R \\ -C_1 & -D_1C_{p2} & -D_1D_{p22}C_R & -D_1D_{p21} & D_1 & -D_1 & -D_1D_{p22}D_R \\ 0 & 0 & 0 & 0 & 0 & 0 & D_2 \\ \hdashline 0 & -C_{p2} & -D_{p22}C_R & -D_{p21} & I & -I & -D_{p22}D_R \end{array}\right]$$

where $W_i(s) = \left[\begin{array}{c|c} A_i & B_i \\ \hline C_i & D_i \end{array}\right]$, $i = 1,2,3$, are state space realizations of weighting matrices.

With the state space realization of G(s) available, we are ready to use μ-synthesis to design a robust controller K(s).

6.4 Control Law Design

Robust controller design is performed on the system shown in Fig.6.3, with the following uncertainty structure: $\Delta = \text{diag}\{\Delta_1, \Delta_2\}$, where $\Delta_1 = \text{diag}\{\delta_1, \delta_2, \delta_3\}$ corresponding to plant uncertainties and Δ_2 is a 2x2 full matrix corresponding to the performance. As we mentioned earlier, the performance is evaluated by H^∞ norm of the transfer function from the command and noise to the tracking error and control constraint. The objective is to design a controller K(s) such that the closed loop system is internally stable and the H^∞ norm of T_{zw} remains small for all the uncertainties considered.

With the procedure illustrated in Section 6.2, we first obtained the optimal H^∞ controller $K_1(s)$ with the optimal H^∞ norm being 7.64. Since $K_1(s)$ ignores the structure

information of Δ and treats Δ as a full matrix, it gives a conservative solution to the problem. Fig.6.8 shows the μ plot and $\bar{\sigma}$ plot of the closed loop system. The difference of them can be seen.

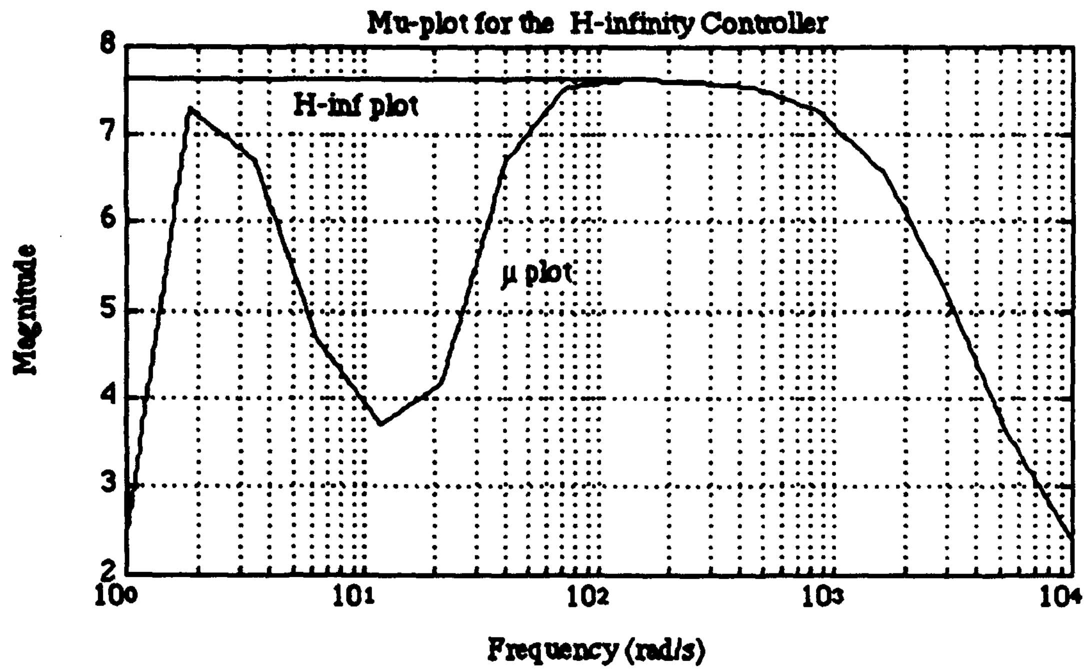


Fig.6.8

As we expected, μ plot is lower than $\bar{\sigma}$ plot, or expressed alternatively $1/\bar{\sigma} \leq 1/\mu$ at each frequency. This implies that the allowable set of uncertainties considered (structured) is larger than that of unstructured uncertainties.

Next we will continue the D-K iteration design procedure described in section 6.2. After two iterations, the process converges to a controller $K_{\mu}(s)$ which minimizes the μ of the closed loop system, see Fig.6.9 for reference.

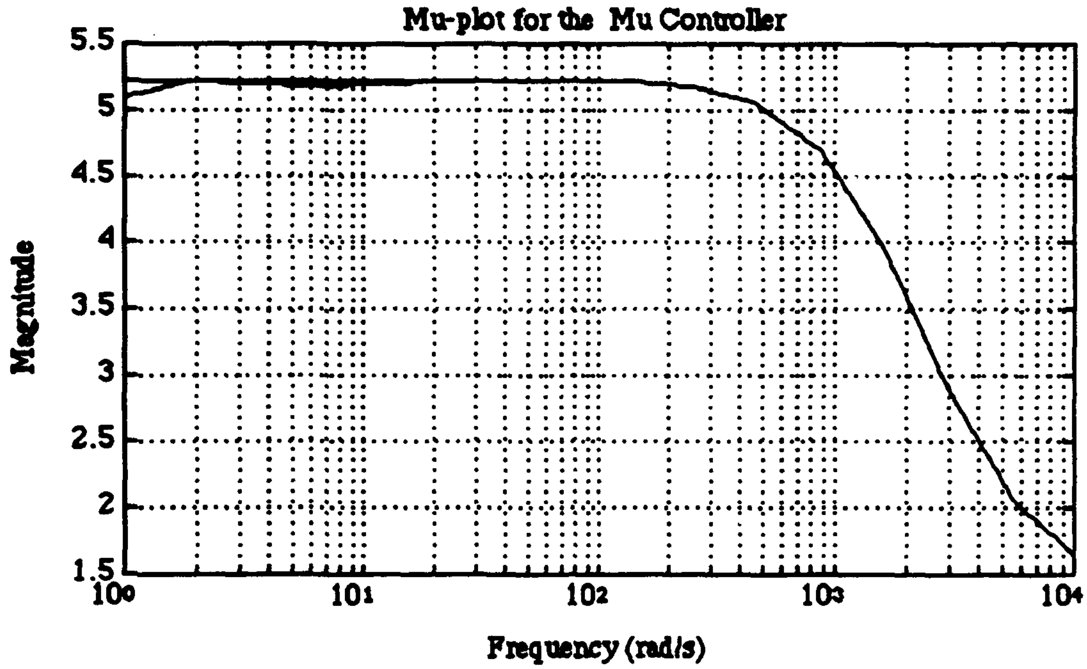


Fig.6.9

In the figure, the lower curve is the μ plot and the upper one the $\bar{\sigma}$ plot of the D-scaled closed loop system. Compare with Fig.6.8, one can see that $K_{\mu}(s)$ gives a smaller μ (= 5.23) than $K_1(s)$ did.

In the following, we will perform some analyses on the original closed loop system shown in Fig. 6.4 with K_{μ} as the controller.

Step responses

The following are the step responses of angle of attack and the corresponding angle of thrust vector nozzle (control input).

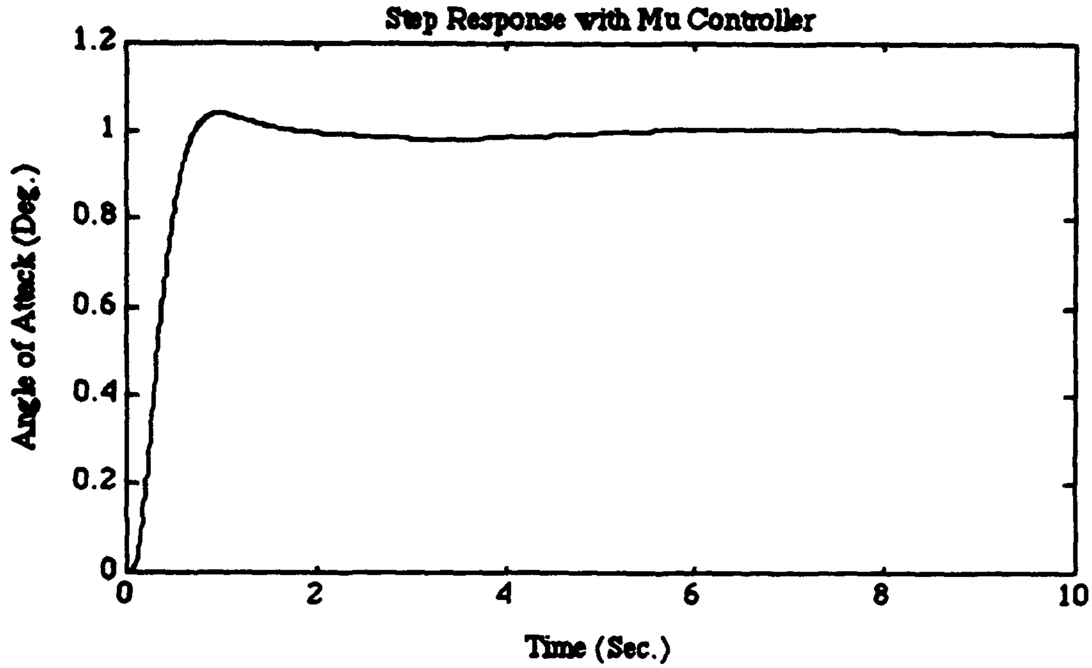


Fig.6.10a

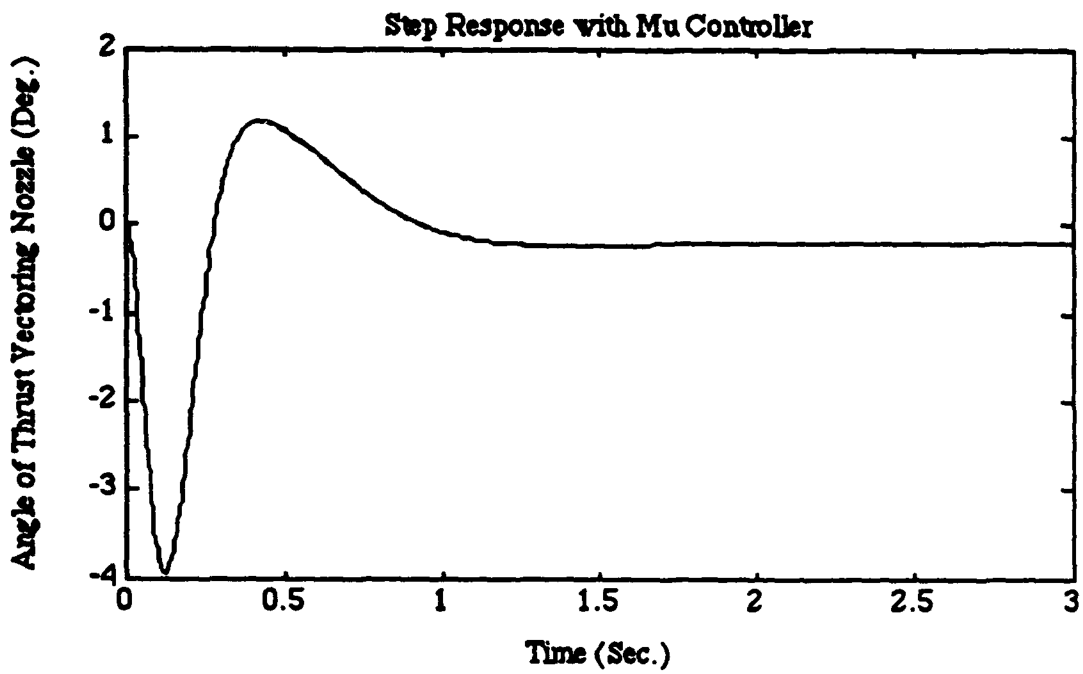


Fig.6.10b

From the design specifications, it is required that the settling time of the response should be $\leq \frac{4.6}{\omega_n \zeta} = 4.6$ second. Observing the above two plots, it can be seen that the design specifications are satisfied.

Perturbation Test

By adding 25% perturbations on M_a, M_q, and M_v, we have the following step responses.

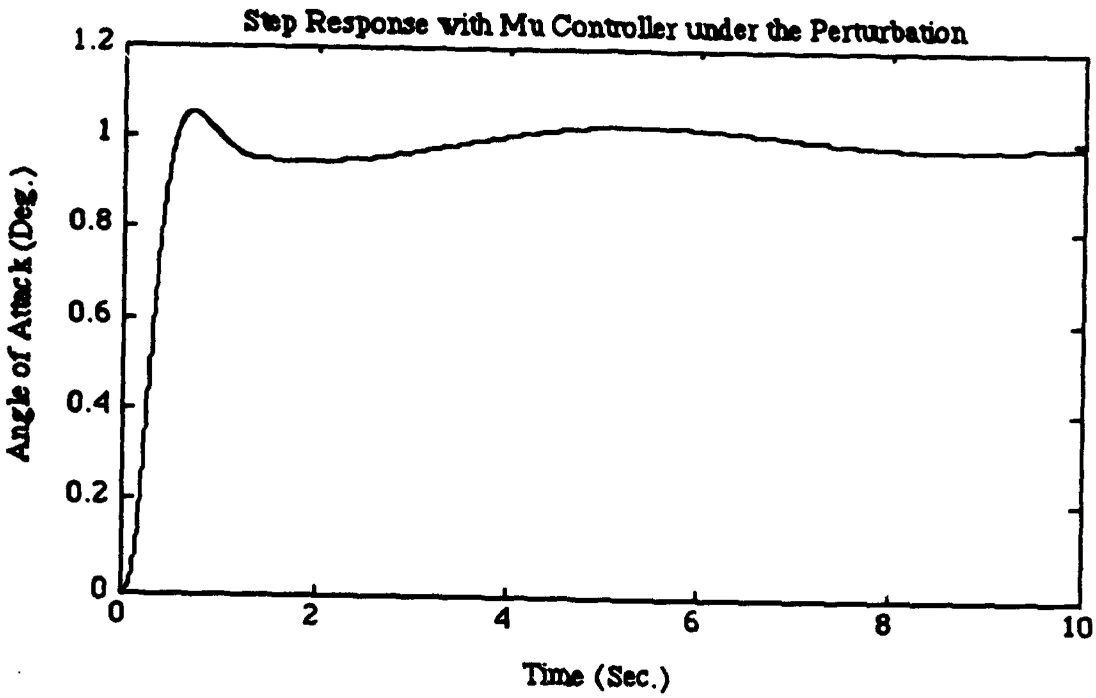


Fig.6.11a

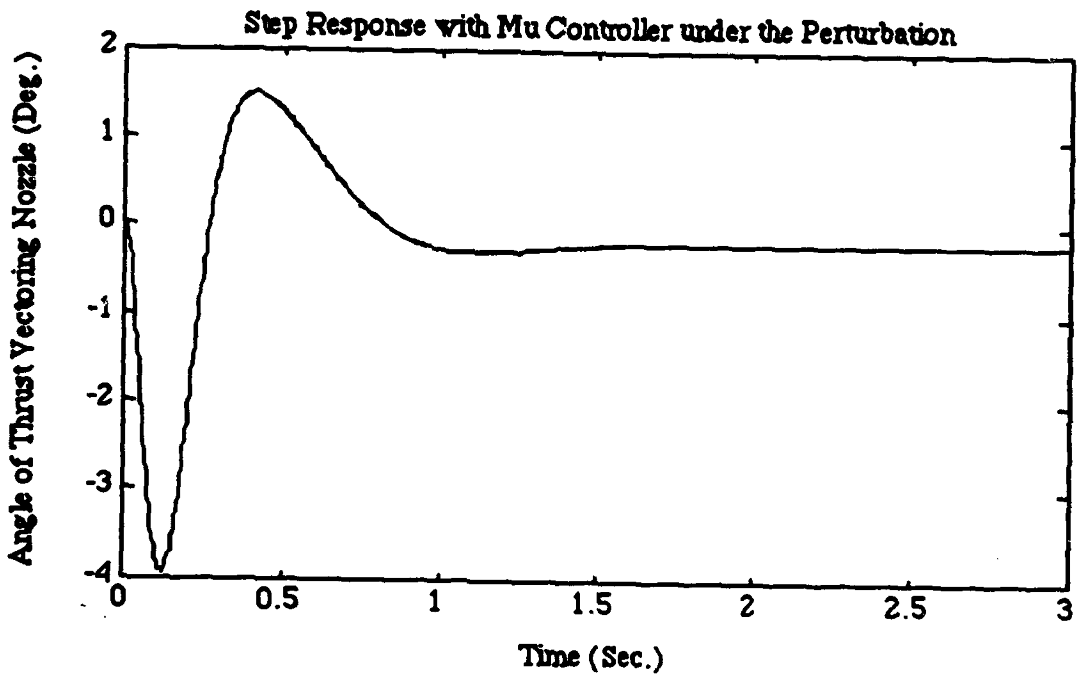


Fig.6.11b

As it shows, the system stability and performance are still satisfactory under the perturbations.

Controller model reduction

High order controller is usually obtained due to the need of "curve fitting" for D(s) in the D-K iteration design procedure. In this example, the Δ has four blocks consisting of three 1x1 and one 2x2 block. Therefore D(s) has n - 1 = 4 - 1 = 3 elements to fit [4]. We choose the three elements with third order, second order and third order rational, proper stable minimal phase functions in the last iteration respectively, which yields an eighth order D(s). With the D(s), inverse of D(s) and the original fifth order plant, the final controller is of twenty-first order. Therefore, the controller should be reduced to a reasonable order. To this end we look at the controllability and observability of $K_\mu(s)$ to see whether we can reduce it to a minimal realization. By finding a balanced realization, K_μ can be reduced to a fifth order controller $K_r(s)$ according to its Hankel singular values [28]:

$$A_c = \begin{bmatrix} -1.2708e{+}03 & -4.5851e{+}00 & -5.7124e{+}01 & 2.1467e{+}02 & -4.6679e{-}01 \\ 0 & -1.8591e{+}01 & -2.2013e{+}01 & -2.3381e{+}00 & -2.0500e{-}02 \\ 0 & 1.2911e{+}01 & -3.8605e{+}00 & 2.5750e{+}00 & -1.5782e{-}02 \\ 0 & 0 & 0 & -9.3984e{+}00 & 3.7513e{-}02 \\ 0 & 0 & 0 & 0 & -1.0001e{-}03 \end{bmatrix}$$

$$B_c = \begin{bmatrix} 9.4612e{+}01 \\ 2.6817e{+}00 \\ 3.6647e{+}00 \\ -7.4246e{+}00 \\ 7.7460e{-}01 \end{bmatrix},$$

$$C_c = [\ -9.4590e{+}01 \quad -3.1892e{+}00 \quad -2.1391e{+}00 \quad 8.0623e{+}00 \quad -7.7481e{-}01\ ];$$

$$D_c = -6.6522e{-}03.$$

With the reduced order controller $K_r(s)$, we have the following μ and $\bar{\sigma}$ plots of the closed loop system.

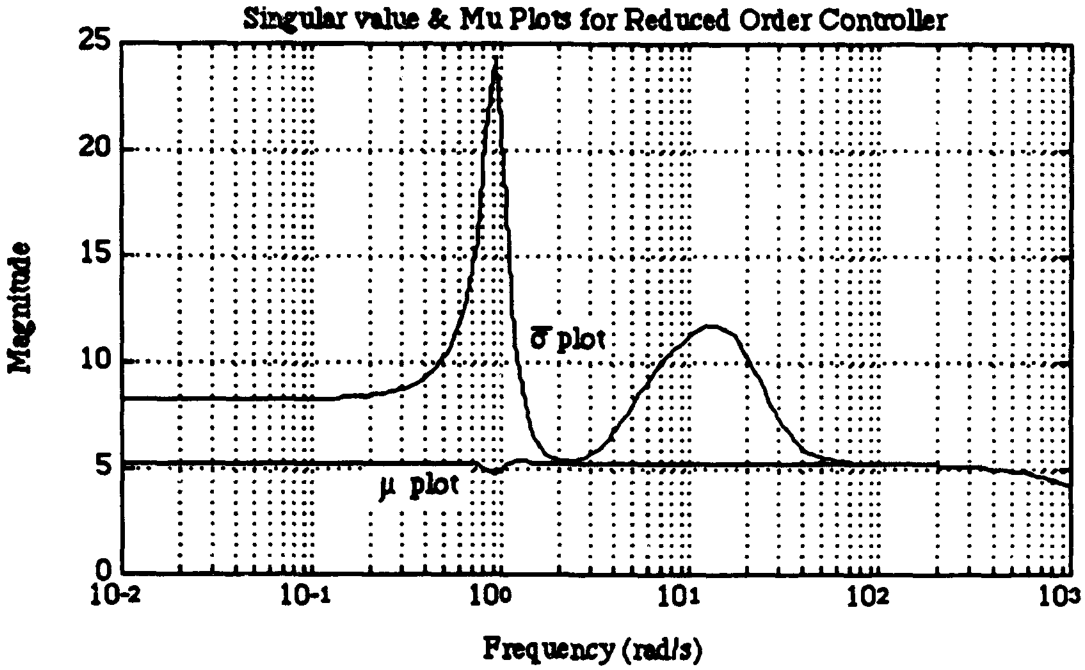


Fig.6.12

As it shows, the reduced order controller $K_r(s)$ maintains almost the same μ for the closed loop system as the full order controller does. Hence robust stability and robust performance are achieved with $K_r(s)$. It can be verified by the following step responses with and without 25% perturbations.

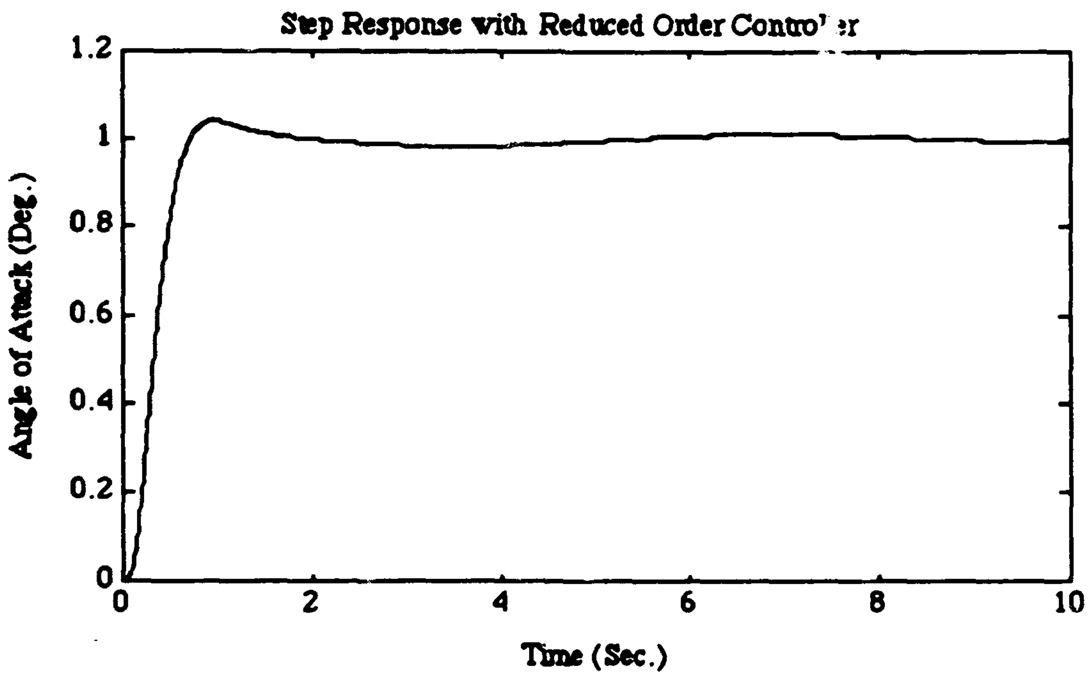


Fig.6.13a

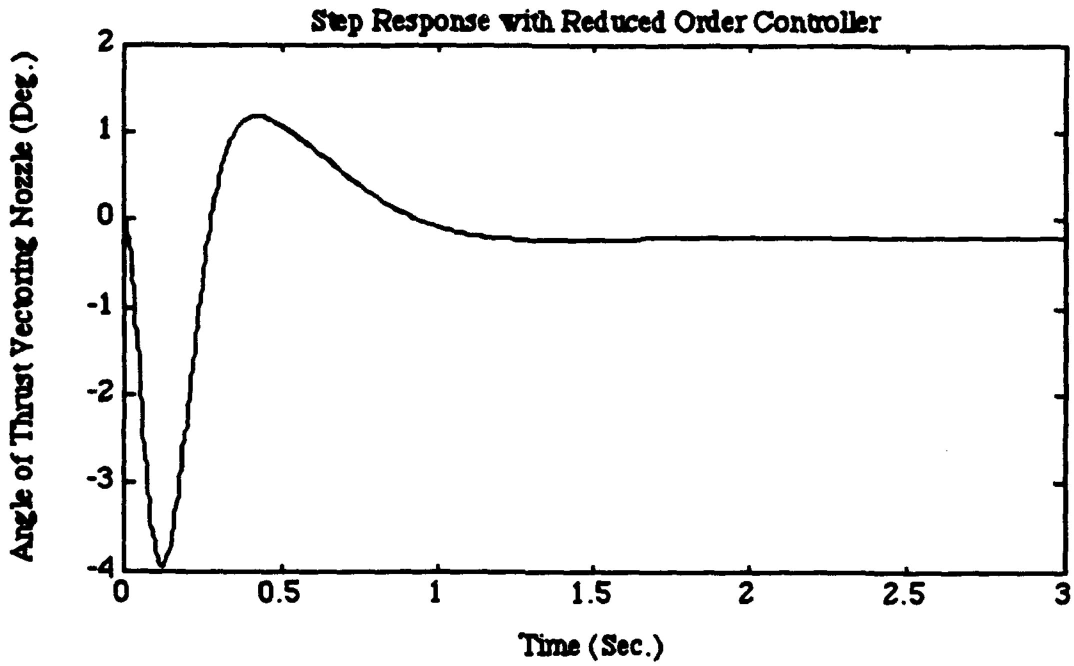


Fig.6.13b

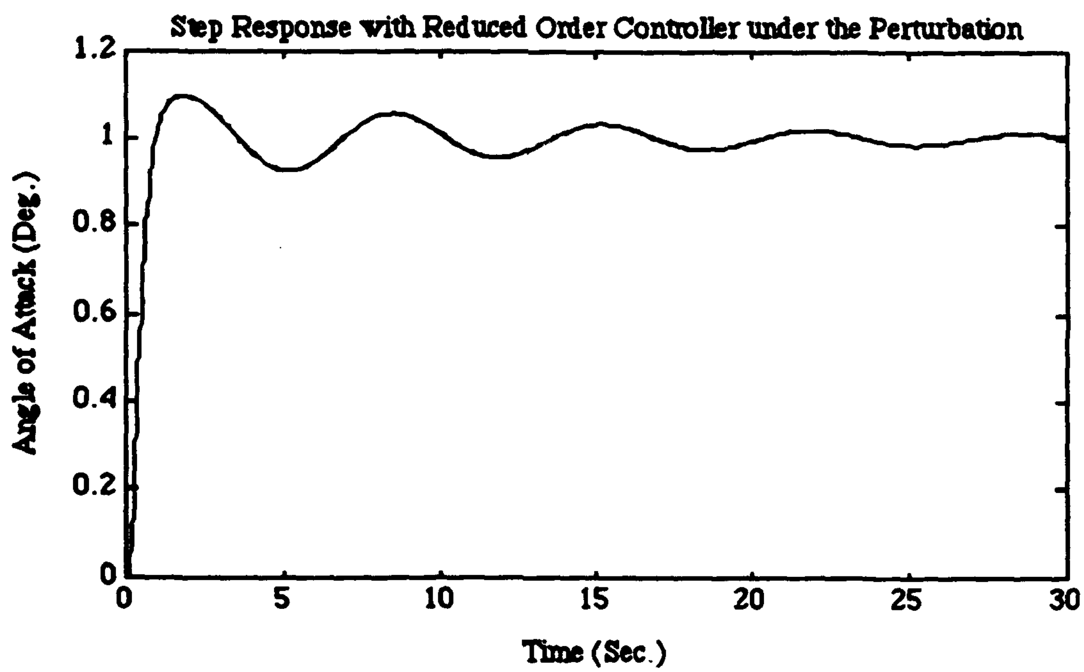


Fig.6.14a

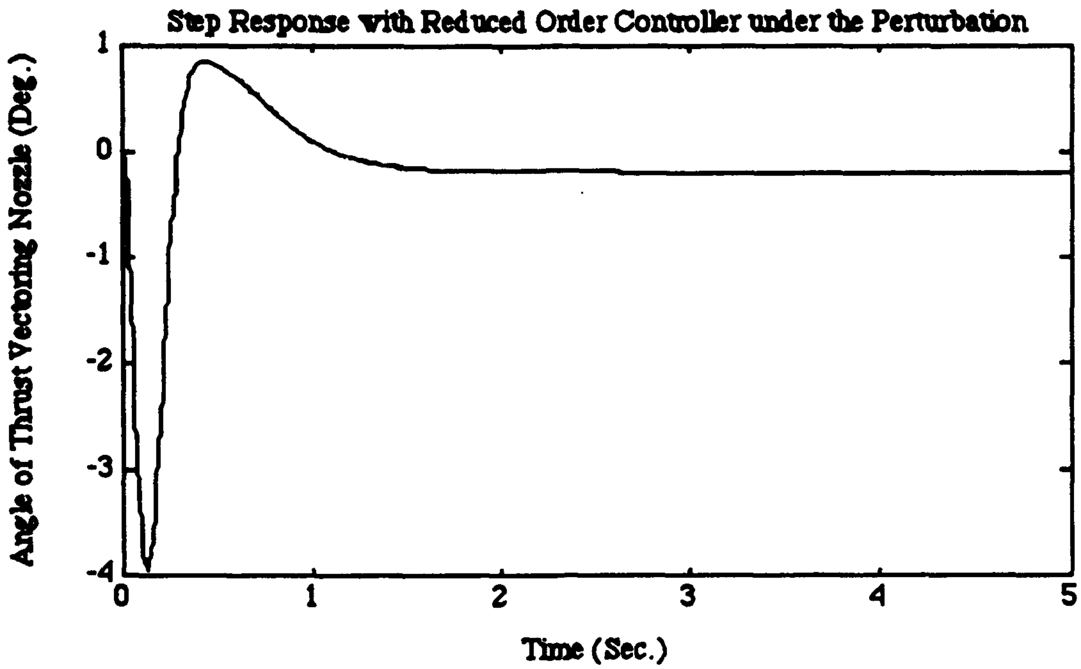


Fig.6.14b

6.5 Conclusions

A fighter aircraft control design problem was considered in this chapter. The design procedure illustrates how to apply μ-synthesis to a real engineering problem. We addressed the issues of problem formulation, D-K iteration, and controller reduction. With the controllers obtained, simulations on system performance and stability analyses were also performed.

CHAPTER 7

CONTROLLER REDUCTION BY STRUCTURED TRUNCATION

Simple controllers are normally preferred over complex controllers because of the less computational requirement and less effort in implementation. The methods for designing low-order controllers can broadly be classified into two categories: *fixed order controller design* and *controller reduction*.

The philosophy in fixed order controller design is to seek to minimize a performance index subject to the constraint that the controller be of fixed degree [29,30]. For controller reduction, one can simply apply model reduction techniques to either the plant model or the controller. There are now at least three rather popular state-space based model reduction techniques, namely, truncation of the internally balanced realization [31,32,33], Hankel-norm approximation [28,34], q-covariance equivalent realization (q-COVER) [63], and coprime factorization method [35]. It is well known that the controller approximation is better than the plant approximation [36]. However, to find a reduced-order controller which approximates the original controllers is not our direct objective. What we seek is to find a reduced-order controller such that the reduced closed-loop system approximates the original closed-loop system.

Jonckheere's work [37] was based on this consideration, in which two Riccati equations are balanced and truncation is carried out with respect to LQG characteristic values. But the relationship between the LQG characteristic values and the closed-loop system properties is unclear. So Jonckheere's approach usually cannot provide satisfactory reduced-order controllers.

In this chapter, we propose a new controller reduction approach which is based on closed-loop considerations other than controller approximations or plant model approximations. In Section 7.1, a method of structured truncation based on the closed-loop properties is developed, and some interesting properties of the controllability and observability gramians of the H_2 optimal closed-loop system are also presented. Some illustrative examples are given in Section 7.2.

7.1 Structured Truncation Approach

Given a closed-loop system, the structured truncation approach is to reduce the order of controller by minimizing a cost function based on the closed-loop system. The cost function will be elaborated later. Although this approach can be applied to any desired closed-loop system, we will concentrate on the optimal H_2 closed-loop system for the simplicity of presentation.

It is well known that the input/output relationship of a system is well represented by its controllability and observability gramians. The balanced truncation [31] is based on these gramians. In the proposed structured truncation approach we also use them. For convenience, some basic knowledge about controllability and observability gramians is reviewed in the following.

Controllability and Observability Gramians

Consider a linear time-invariant system

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

i.e., the transfer function is

$$G(s) := \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = D + C(sI - A)^{-1}B$$

The eigenvalues of A are assumed to be strictly in the left half-plane and the controllability gramian P and the observability gramian Q are defined as

$$P := \int_0^\infty \exp(At)BB'\exp(A't)dt$$

$$Q := \int_0^\infty \exp(A't)C'C\exp(At)dt$$

where A' is the transpose of A. P and Q satisfy the following Lyapunov equations

$$AP + PA' + BB' = 0$$
$$A'Q + QA + C'C = 0$$

and have the following properties.

Property 7.1

If all the eigenvalues of A are strictly in the left-half plane, then

(a) $P > 0$ if and only if (A, B) is completely controllable.

(b) $Q > 0$ if and only if (A, C) is completely observable.

If the state-space coordinates of the system are changed to $z = T^{-1}x$ for some non-singular T then

$$\dot{z} = T^{-1}ATz + T^{-1}B,\ \ y = CTz + Du$$

Furthermore, the controllability and observability gramians become $T^{-1}PT'^{-1}$ and $T'QT$ respectively, and the product PQ will be transformed to $T^{-1}PQT$. Therefore the eigenvalues of PQ are invariant under state-space transformations, and are input/output invariant. A useful state-space realization in this respect is the balanced realization where $P = Q = \mathrm{diag}(\sigma_1, \sigma_2, ..., \sigma_n)$, therefore $PQ = \mathrm{diag}(\sigma_1^2, ..., \sigma_n^2)$.

Structured Truncation

Consider the block diagram of in Fig. 7.1.

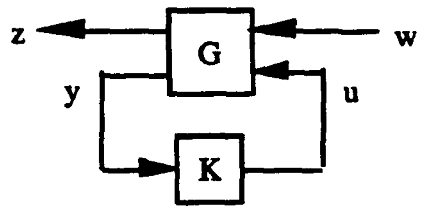


Fig. 7.1 A generalized plant G with a controller K.

G is a generalized nominal plant, w contains all external inputs, including disturbances, sensor noises, and commands; the output z is the controlled output; y is the measured output; and u is the control input. A state-space realization of the generalized plant G is given as

$$G(s) := \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] \tag{7-1}$$

where $A \in \mathbb{R}^{n\times n}$, $B_1 \in \mathbb{R}^{n\times m_1}$, $B_2 \in \mathbb{R}^{n\times m_2}$, $C_1 \in \mathbb{R}^{p_1\times n}$, and $C_2 \in \mathbb{R}^{p_2\times n}$. The controller K is assumed to be

$$K(s) = \left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & D_c \end{array}\right] \tag{7-2}$$

where $A_c \in \mathbf{R}^{n_c \times n_c}$, $B_c \in \mathbf{R}^{n_c \times p_2}$, $C_c \in \mathbf{R}^{m_2 \times n_c}$, and $D_c \in \mathbf{R}^{m_2 \times p_2}$. T_{zw} denotes the closed-loop transfer function from w to z.

The objective is to reduce the order of the controller with which the closed-loop system approximates the original closed-loop system. Hence we consider the controllability and observability gramians of the closed-loop transfer function T_{zw}. The balancing technique can not be applied on the gramians of the closed-loop system because the balancing transformation takes place in the whole state space of the nominal plant and the controller. With the states of the closed-loop system arranged as $[x_p', x_c']'$ where x_p is the set of the n states of the plant and x_c is the set of the n_c states from the controller, to avoid mixing up the states from the plant and the controller, only the following structured transformation

$$T = \begin{bmatrix} T_n & 0 \\ 0 & T_{n_c} \end{bmatrix} \tag{7-3}$$

is allowed. Without loss of generality we use the transformation shown below

$$T = \begin{bmatrix} I_n & 0 \\ 0 & T_{n_c} \end{bmatrix} \tag{7-4}$$

Let P and Q be the closed-loop controllability and observability gramians respectively and define J = PQ. If the closed-loop system can be balanced, i.e. J can be diagonalized, then the states corresponding to the small diagonal elements can be truncated from the system. Due to the structure restriction on transformation, it is impossible to totally balance the closed-loop system.

By applying the transformation T of (7-4) to the closed-loop system, J becomes

$$\begin{bmatrix} I_n & 0 \\ 0 & T_{n_c}^{-1} \end{bmatrix} J \begin{bmatrix} I_n & 0 \\ 0 & T_{n_c} \end{bmatrix}$$

$$= \begin{bmatrix} I_n & 0 \\ 0 & T_{n_c}^{-1} \end{bmatrix} \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \begin{bmatrix} I_n & 0 \\ 0 & T_{n_c} \end{bmatrix}$$

$$= \begin{bmatrix} J_{11} & J_{12} T_{n_c} \\ T_{n_c}^{-1} J_{21} & T_{n_c}^{-1} J_{22} T_{n_c} \end{bmatrix}$$

Let $T_{n_c}^{-1} = \begin{bmatrix} S_1' \\ \underline{s}' \end{bmatrix}$ and $T_{n_c} = \begin{bmatrix} T_1 & \underline{t} \end{bmatrix}$, where $\underline{s}$ and $\underline{t} \in \mathbf{R}^{nc}$, so

$$\begin{bmatrix} J_{11} & J_{12} T_{n_c} \\ T_{n_c}^{-1} J_{21} & T_{n_c}^{-1} J_{22} T_{n_c} \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} T_1 & J_{12} \underline{t} \\ S_1' J_{21} & S_1' J_{22} T_1 & S_1' J_{22} \underline{t} \\ \underline{s}' J_{21} & \underline{s}' J_{22} T_1 & \underline{s}' J_{22} \underline{t} \end{bmatrix}$$

The problem is proposed to find T_{n_c} such that

$$f(T_{n_c}) = \left\| \begin{bmatrix} & 0 & J_{12} \underline{t} \\ & & S_1' J_{22} \underline{t} \\ \underline{s}' J_{21} & \underline{s}' J_{22} T_1 & \underline{s}' J_{22} \underline{t} \end{bmatrix} \right\|_F \qquad (7\text{-}5)$$

is minimized.

Here $\|.\|_F$ denotes the Frobenius-norm. Then we truncate the states of the controller corresponding to the small cost defined in (7-5).

For the H_2 optimization problem with the generalized plant

$$G(s) := \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array} \right]$$

and the optimal controller

$$K_{opt}(s) := \left[\begin{array}{c|c} A_c & -L_2 \\ \hline F_2 & 0 \end{array}\right]$$

where $F_2 := -B_2'X_2$, $L_2 := -Y_2C_2'$ and $A_c := A + B_2F_2 + L_2C_2$, and X_2 and Y_2 are the solutions of the following two Riccati equations

$$A'X_2 + X_2A - X_2B_2B_2'X_2 + C_1'C_1 = 0$$

$$AY_2 + Y_2A' - Y_2C_2'C_2Y_2 + B_1B_1' = 0.$$

The closed-loop system with the optimal H_2 controller is of the form

$$\mathcal{G}(s) := \left[\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline \mathcal{C} & 0 \end{array}\right] = \left[\begin{array}{cc|c} A & B_2F_2 & B_1 \\ -L_2C_2 & A + B_2F_2 + L_2C_2 & -D_{21}L_2 \\ \hline C_1 & F_2D_{12} & 0 \end{array}\right]$$

Let P and Q be the controllability and observability gramians of $\mathcal{G}(s)$ respectively, then we have the following proposition.

Proposition 7.1: P and Q are of form

$$P = \begin{bmatrix} P_2 + X_2 & P_2 \\ P_2 & P_2 \end{bmatrix} \qquad Q = \begin{bmatrix} Q_2 + Y_2 & -Q_2 \\ -Q_2 & Q_2 \end{bmatrix}$$

where P_2 and Q_2 satisfy the following two Lyapunov equations

$$(A + B_2F_2)P_2 + P_2(A + B_2F_2)' + L_2L_2' = 0 \tag{7-6}$$

$$(A + L_2C_2)'Q_2 + Q_2(A + L_2C_2) + F_2'F_2 = 0 \tag{7-7}$$

This result can be easily verified.

So the structures guarantee

$$J = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & 0 \end{bmatrix}$$

The problem mentioned above turns out to find T_{n_c} such that

$$\left\| \begin{bmatrix} 0 & J_{12}\underline{t} \\ \underline{s}'J_{21} & 0 \end{bmatrix} \right\|_F$$

or $f(T_{n_c}) = \underline{s}'J_{21}'J_{21}\underline{s} + \underline{t}'J_{12}'J_{12}\underline{t}$ is minimized.

Theorem 7.1

There exist two vectors $\underline{s}$ and $\underline{t}$ such that $\underline{s}'\underline{t} = 1$,

$$f(\underline{s}, \underline{t}) = \underline{s}'J_{21}'J_{21}\underline{s} + \underline{t}'J_{12}'J_{12}\underline{t}$$

is minimized.

Proof: Let

$$f(\underline{s}, \underline{t}) = \underline{s}'J_{21}'J_{21}\underline{s} + \underline{t}'J_{12}'J_{12}\underline{t} - 2\lambda(\underline{s}'\underline{t} - 1)$$

then set

$$\frac{\partial f(\underline{s}, \underline{t})}{\partial \underline{s}} = 2J_{21}'J_{21}\underline{s} - 2\lambda\underline{t} = 0 \tag{7-8}$$

$$\frac{\partial f(\underline{s}, \underline{t})}{\partial \underline{t}} = 2J_{12}'J_{12}\underline{t} - 2\lambda\underline{s} = 0 \tag{7-9}$$

$$\underline{s}'\underline{t} = 1.$$

It is easy to see that λ is the eigenvalue of the matrix

$$\begin{bmatrix} 0 & J_{12}'J_{12} \\ J_{21}'J_{21} & 0 \end{bmatrix} \tag{7-10}$$

and is $\begin{bmatrix} \underline{s} \\ \underline{t} \end{bmatrix}$ the eigenvector corresponding to the smallest eigenvalue of (10.10). Q.E.D.

The matrix (7-10) is a special case of Hamiltonian and has the following property.

Property 7.2

If both J_{12} and J_{21} are full of rank then the eigenvalues of (7-10) are real.

Proof:

From (7-8) and (7-9)

$$J_{21}'J_{21}\underline{s} = \lambda\underline{t} \tag{7-11}$$

$$J_{12}'J_{12}\underline{t} = \lambda\underline{s} \tag{7-12}$$

Multiply (7-11) by $\underline{s}'$ and (7-12) by $\underline{t}'$, and plus them together. Here $\underline{s}'$ means the conjugate transpose of $\underline{s}$.

$$\underline{s}'J_{21}'J_{21}\underline{s} + \underline{t}'J_{12}'J_{12}\underline{t} = \lambda(\underline{s}'\underline{t} + \underline{t}'\underline{s}) \tag{7-13}$$

From (7-13) we see that $\mathrm{Im}(\underline{s}'J_{21}'J_{21}\underline{s}+\underline{t}'J_{12}'J_{12}\underline{t}) = 0$, and $\mathrm{Im}(\underline{s}'\underline{t}+\underline{t}'\underline{s}) = 0$. So $\mathrm{Im}(\lambda) = 0$. Q.E.D.

7.2 Examples

Two examples are shown to illustrate the proposed structured truncation approach for controller truncation.

Example 7.1:

Given a plant:

$$G(s) := \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right] = \left[\begin{array}{cc|cc|c} -1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 \\ \hline 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \hline 1 & 1 & 0 & 1 & 0 \end{array}\right]$$

Full-order optimal H_2 controller is:

$$\left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & 0 \end{array}\right] = \left[\begin{array}{cc|c} -0.8619 & 0.1381 & -0.1381 \\ -4.8613 & -6.7884 & 4.5523 \\ \hline -0.3090 & -4.2361 & 0 \end{array}\right]$$

By the structured truncation, the controller can be reduced to the following 1st-order controller

$$\left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & 0 \end{array}\right] = \left[\begin{array}{c|c} -6.4397 & -3.4225 \\ \hline 5.5247 & 0 \end{array}\right]$$

In Table.1, H_2-norms of reduced-order closed-loop system and L_∞-error bound between the closed-loop system with the full-order controller and those with reduced-order

controllers obtained by our new structured truncation, balanced truncation, coprime factorization, and optimal projection are listed.

	$\| T_{zw} \|_2^2$	$\| T_{full} - T_{red} \|_\infty$
Full-order Optimal Controller	91.8196689	0
1st-order controllers		
structured Truncation	91.857999	0.233720
Balanced truncation	91.871408	0.392831
coprime factorization	91.870133	0.361891
Optimal Projection	91.856689	0.269276

Table 1.

Example 7.2:

Given plant

$$G(s) := \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right] = \left[\begin{array}{ccc|cccc|c} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & -2 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & -3 & 0 & 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hdashline 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

The full-order optimal H_2 controller is:

$$\left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & 0 \end{array}\right] = \left[\begin{array}{ccc|c} -4.1899 & -2.6639 & -2.6667 & 2.5950 \\ -2.6639 & -2.1379 & -0.1407 & 0.0689 \\ -2.6667 & -0.1407 & -3.1435 & 0.0717 \\ \hline -2.5950 & -0.0689 & -0.0717 & 0 \end{array}\right]$$

The controller is unstable.

The reduced second and first-order controllers by the structured truncation are

$$\left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & 0 \end{array}\right] = \left[\begin{array}{cc|c} -1.3875 & -3.6269 & 0.0263 \\ -2.7605 & -5.2019 & -2.5453 \\ \hline -0.6207 & 2.6198 & 0 \end{array}\right]$$

and

$$\left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & 0 \end{array}\right] = \left[\begin{array}{c|c} 0.2507 & 1.6406 \\ \hline -2.0529 & 0 \end{array}\right]$$

The comparison of the proposed structured truncation with the others is shown in Table 2.

	$\lVert Tzw \rVert_2^2$	$\lVert T_{full} - T_{red} \rVert_\infty$
Full-order Optimal Controller	22.490551	0
2nd-order controllers		
Structured Truncation	22.491118	0.019268
Balanced truncation	(The controller is unstable)	
Coprime factorization	22.491945	0.051115
Optimal Projection	22.491114	0.021112
1st-order controllers		
Structured Truncation	33.797984	4.913441
Balanced truncation	(The controller is unstable)	
Coprime factorization	42.666848	16.386789
Optimal Projection	30.333357	4.4430115

Table 2.

The optimal projection is the fixed-order controller design to minimize the H_2-norm of the closed-loop system. For the two examples, the H_2-norm of the closed-loop system with the reduced-order controller by the new structured truncation is very close to the result of optimal projection design. The structured truncation also results in small L_∞-error bounds.

CHAPTER 8

A PARAMETRIZATION APPROACH TO REDUCED-ORDER H^∞ CONTROLLER DESIGN

8.1 Introduction

By the state-space approach [7] to H^∞ optimization, an (sub)optimal controller can be easily obtained and the order of the controller is not higher than that of the generalized plant. However, in many engineering problems, the order of generalized plant can be very high. This is due to the fact that the generalized plant consists of the original plant as well as all the weighting matrices which are chosen to meet certain design specifications. Hence, the order of the H^∞ controller obtained by the standard state-space approach is usually too high to be implemented in practice, and therefore a systematical methodology of designing a reduced-order H^∞ robust controller is desired.

This issue has attracted considerable attention in the past few years, and several approaches have been proposed to address this problem. Bernstein *et al.* [30,38] proposed a fixed order controller design method based on the optimal projection theory. It was shown that an optimal reduced-order H^2/H^∞ controller can be obtained by solving several coupled Riccati equations. However, the computation involved is very complicated. A direct design method based on the well-known Bounded Real Lemma [39] was proposed by Hsu *et al.* [40], where a set of sufficient conditions and design algorithms were derived for a reduced-order H^∞ controller design. Chang *et al.* [41,42] considered observer-based controller parametrization, and pointed out that by selecting a suitable parameter matrix, one can make the realization of the stabilizing controller non-minimal, whose order then can be reduced. Choi *et al.* in [43] proposed a method of constructing such a suitable parameter matrix in order to get a reduced-order stabilizing controller.

These results, especially the work by Choi *et al.* [43], motivated our work in designing reduced-order H^∞ controllers. In this chapter we will present a direct design algorithm for $(n-p_2)$th order H^∞ controller, where n is the order of the generalized controller and p_2 is the number of independent measured outputs. The major idea is to select a parameter matrix such that the realization of the controller obtained from DGKF formulas

[7] is not minimal. By deleting the unobservable states of the controller, $(n-p_2)$th order H^∞ controller can be obtained. The content of this chapter is arranged as follows. Section 8.2 is a preliminary section which provides the standard state space approach to H^∞ optimization problem, including the formula of all stabilizing (sub)optimal controller. We present our main result in section 8.3, where a set of formulas are given for constructing a reduced order H^∞ controller. Section 8.4 shows an illustrative example and Section 8.5 is a conclusion.

8.2 Preliminaries

The following block diagram is the standard H^∞ optimization configuration.

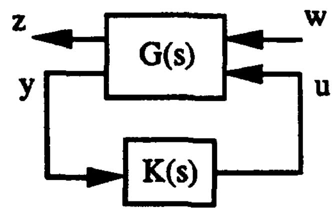


Fig. 8.1 The standard block diagram for H^∞ optimization

The G(s) is the generalized plant, including the original plant and possible weighting matrices. The signals z, y, w, and u are the controlled output, the measured output, the exogenous input and the control input respectively. The standard H^∞ optimization problem is: To design a proper controller K(s) such that the closed-loop system is internally stable and $\|T_{zw}(s)\|_\infty$ is minimized, where $T_{zw}(s)$ denotes the transfer function of the closed loop system from w to z.

In the DGKF approach [7], the realization of the generalized plant G(s) is assumed to be

$$G(s) = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right] \tag{8-1}$$

where G(s) is partitioned such that $G_{11}(s)$ is $p_1 \times m_1$, $G_{12}(s)$ is $p_1 \times m_2$, $G_{21}(s)$ is $p_2 \times m_1$, and $G_{22}(s)$ is $p_2 \times m_2$. It is also assumed that:

(i) Both $G_{12}(s)$ and $G_{21}(s)$ do not have any transmission zeros on the $j\omega$-axis.

(ii) (A, B_2) is stabilizable and (C_2, A) is detectable.

(iii) $D_{12}'[C_1 \; D_{12}] = [0 \;\; I]$

(iv) $\begin{bmatrix} B_1 \\ D_{21} \end{bmatrix} D_{21}' = \begin{bmatrix} 0 \\ I \end{bmatrix}$

The two Riccati equations involved are:

$$A'X_\infty + X_\infty A + X_\infty(\gamma^{-2}B_1B_1' - B_2B_2')X_\infty + C_1'C_1 = 0 \tag{8-2}$$

and

$$AY_\infty + Y_\infty A' + Y_\infty(\gamma^{-2}C_1'C_1 - C_2'C_2)Y_\infty + B_1B_1' = 0 \tag{8-3}$$

The following famous theorem characterizes all (sub)optimal stabilizing controllers such that $\|T_{zw}\|_\infty < \gamma$.

Theorem 8.1 [7]

For a given γ, there exists a stabilizing controller such that $\|T_{zw}\|_\infty<\gamma$ if and only if the following three conditions hold:

(i) There exists a positive semidefinite stabilizing sc ition $X_\infty(\gamma)$ to (8-2).

(ii) There exists a positive semidefinite stabilizing solution $Y_\infty(\gamma)$ to (8-3).

(iii) $\rho[X_\infty(\gamma)Y_\infty(\gamma)] < \gamma^2$, where ρ denotes the spectral radius of a matrix.

When these conditions hold, all stabilizing (sub)optimal controllers K(s) can be parametrized by a stable proper parameter matrix Q(s) whose H^∞ norm is less than γ, as shown in the following figure.

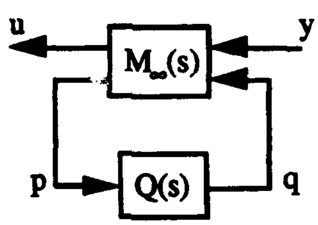


Fig. 8.2 The parametrization of all stabilizing (sub)optimal controllers

Here
$$M_\infty(s) = \left[\begin{array}{c|cc} \hat{A}_\infty & -Z_\infty L_\infty & Z_\infty B_2 \\ \hline F_\infty & 0 & I \\ -C_2 & I & 0 \end{array}\right], \tag{8-4}$$

with
$$\hat{A}_\infty = A + \gamma^{-2}B_1B_1'X_\infty + B_2F_\infty + Z_\infty L_\infty C_2 \;, \tag{8-5}$$

$$F_\infty = -B_2'X_\infty \tag{8-6}$$

$$L_\infty = -Y_\infty C_2' \tag{8-7}$$

$$Z_\infty = (I - \gamma^{-2} Y_\infty X_\infty)^{-1} \tag{8-8}$$

and

$$Q(s) = \left[\begin{array}{c|c} A_q & B_q \\ \hline C_q & D_q \end{array}\right]. \tag{8-9}$$

8.3 Reduced-Order H^∞ Controller Design

In this section, we will show the existence of an $(n-p_2)$th stabilizing H^∞ sub-optimal controller, where n is the order of the generalized plant and p_2 is the number of independent measured outputs. By using the DGKF's formulas in Theorem 8.1, the state-space equations for sub-optimal controller K(s) can be expressed as

$$\dot{x} = \hat{A}_\infty x - Z_\infty L_\infty y + Z_\infty B_2 q \tag{8-10}$$

$$u = F_\infty x + q \tag{8-11}$$

$$p = -C_2 x + y \tag{8-12}$$

$$\dot{\tilde{x}} = A_q \tilde{x} + B_q p \tag{8-13}$$

$$q = C_q \tilde{x} + D_q p \tag{8-14}$$

where the dimension of A_q is $(n-p_2)$. Simplifying the above equations, we have the following state-space representation for the controller,

$$K(s) = \left[\begin{array}{cc|c} A + \gamma^{-2} B_1 B_1' X_\infty + B_2 F_\infty + Z_\infty L_\infty C_2 - Z_\infty B_2 D_q C_2 & Z_\infty B_2 C_q & Z_\infty B_2 D_q - Z_\infty L_\infty \\ -B_q C_2 & A_q & B_q \\ \hline F_\infty - D_q C_2 & C_q & D_q \end{array}\right] \tag{8-15}$$

Note the fact that the realization of K(s) may not be minimal, which depends on the $\{A_q, B_q, C_q, D_q\}$ chosen. This suggests that if one can choose a set of $\{A_q, B_q, C_q, D_q\}$ such that the realization of K(s) in eq.(8-15) is non minimal, then by deleting the unobservable or/and uncontrollable states of K(s), a reduced-order controller can be obtained, and the system performance and stability can be maintained at same time. To this end, we apply a similarity transformation

$$T = \begin{bmatrix} I & 0 \\ -X & I \end{bmatrix} \tag{8-16}$$

to K(s), where X will be determined later. With the transformation, eq.(8-15) becomes

$$K(s) = \left[\begin{array}{c|c} A_M & B_M \\ \hline C_M & D_M \end{array}\right], \tag{8-17}$$

where $$A_M = \begin{bmatrix} A_{M11} & A_{M12} \\ A_{M21} & A_{M22} \end{bmatrix}$$

with

$$A_{M11} = A + \gamma^{-2}B_1B_1'X_\infty + B_2F_\infty + Z_\infty L_\infty C_2 - Z_\infty B_2D_qC_2 - Z_\infty B_2C_qX \tag{8-18}$$

$$A_{M12} = Z_\infty B_2C_q \tag{8-19}$$

$$A_{M21} = X(A + \gamma^{-2}B_1B_1'X_\infty + B_2F_\infty + Z_\infty L_\infty C_2 - Z_\infty B_2D_qC_2)$$
$$- B_qC_2 - XZ_\infty B_2C_qX - A_qX \tag{8-20}$$

$$A_{M22} = XZ_\infty B_2C_q + A_q, \tag{8-21}$$

and

$$B_M = \begin{bmatrix} B_{M1} \\ B_{M2} \end{bmatrix} = \begin{bmatrix} Z_\infty B_2D_q - Z_\infty L_\infty \\ X(Z_\infty B_2D_q - Z_\infty L_\infty) + B_q \end{bmatrix}. \tag{8-22}$$

$$C_M = \begin{bmatrix} C_{M1} & C_{M2} \end{bmatrix} = \begin{bmatrix} F_\infty - D_qC_2 - C_qX & C_q \end{bmatrix} \tag{8-23}$$

$$D_M = D_q \tag{8-24}$$

Note that our objective is to choose an X in the similarity transformation such that

$$C_{M1} = 0 \tag{8-25}$$

$$A_{M21} = 0. \tag{8-26}$$

If this is achieved, the unobservable pair (C_{M1}, A_{M11}) in eq.(8-17) can be deleted and the controller in eq.(8-17) is reduced to

$$K(s) = \left[\begin{array}{c|c} A_{M22} & B_{M2} \\ \hline C_{M2} & D_M \end{array}\right] \tag{8-27}$$

whose order is $r = \dim(A_q) = n\text{-}p_2$. In fact, the equations (8-25) and (8-26) are identical to

$$\begin{bmatrix} C_q & D_q \end{bmatrix}\begin{bmatrix} X \\ C_2 \end{bmatrix} = F_\infty \tag{8-28}$$

$$X\overline{A} - A_qX - B_qC_2 = 0 \tag{8-29}$$

where

$$\overline{A} = A + \gamma^{-2}B_1B_1'X_\infty + B_2F_\infty + Z_\infty L_\infty C_2 - Z_\infty B_2F_\infty \tag{8-30}$$

The discussion of the existence of the X to equations (8-28) and (8-29) will be given in the following, which plays a key role in our main result.

Lemma 8.1 $\overline{A}$ is a stable matrix.

Proof: Define

$$P := \gamma^2 Z_\infty Y_\infty F_\infty' F_\infty - B_2F_\infty + Z_\infty B_2F_\infty, \tag{8-31}$$

then it is easy to verify that

$$P = (-\gamma^2 Z_\infty Y_\infty X_\infty - I + Z_\infty)B_2F_\infty = [Z_\infty(I - \gamma^2 Y_\infty X_\infty) - I]B_2F_\infty = 0. \tag{8-32}$$

Note eq.(8-30) can be expressed as

$$\begin{aligned} \overline{A} &= A + \gamma^{-2}B_1B_1'X_\infty + B_2F_\infty + Z_\infty L_\infty C_2 - Z_\infty B_2F_\infty \\ &= A_{tmp} + Z_\infty Y_\infty(-\gamma^2 F_\infty' F_\infty - C_2'C_2) - P \\ &= A_{tmp} + Z_\infty Y_\infty(-\gamma^2 F_\infty' F_\infty - C_2'C_2), \end{aligned} \tag{8-33}$$

where

$$A_{tmp} = A + \gamma^{-2}B_1B_1'X_\infty. \tag{8-34}$$

It is straightforward to show that [7]

$$T^{-1}J_{tmp}T = J_\infty, \tag{8-35}$$

where

$$J_\infty = \begin{bmatrix} A' & \gamma^2 C_1'C_1 - C_2'C_2 \\ -B_1B_1' & -A \end{bmatrix}, \tag{8-36}$$

$$J_{tmp} = \begin{bmatrix} A_{tmp}' & \gamma^{-2}F_\infty' F_\infty - C_2'C_2 \\ -B_1B_1' & -A_{tmp} \end{bmatrix}, \tag{8-37}$$

and $$T = \begin{bmatrix} I & -\gamma^{-2}X_\infty \\ 0 & I \end{bmatrix}. \tag{8-38}$$

Hence, it is easy to see that the stabilizing solution to eq.(8-37) can be expressed in terms of Y_∞ which is the stabilizing solution to eq.(8-36) (or eq.8-3)), i.e.,

$$Y_{tmp} = Z_\infty Y_\infty \tag{8-39}$$

is the stabilizing solution to Riccati equation eq.(8-37) [7]. Substituting eq.(8-39) into eq.(8-33), we have

$$\overline{A} = A_{tmp} + Y_{tmp}(\,-\gamma^2 F_\infty' F_\infty - C_2' C_2),$$

which implies $\overline{A}$ is stable. Thus we proved the lemma.

Without loss of generality, we assume that

$$C_2 = [\, I_{p2} \;\; 0\,]. \tag{8-40}$$

Then we partition $\overline{A}$ and X compatibly such that

$$\overline{A} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \tag{8-41}$$

$$X = [\, X_1 \;\; X_2 \,]. \tag{8-42}$$

With the partition, eq.(8-29) can be expressed as

$$[\, X_1 \;\; X_2 \,] \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} - \begin{bmatrix} A_q X_1 & A_q X_2 \end{bmatrix} - \begin{bmatrix} B_q & 0 \end{bmatrix} = 0. \tag{8-43}$$

or equivalently

$$X_1 A_{11} + X_2 A_{21} - A_q X_1 = B_q \tag{8-44}$$

$$X_1 A_{12} + X_2 A_{22} - A_q X_2 = 0. \tag{8-45}$$

Lemma 8.2 The pair (A_{12}, A_{22}) is detectable.

Proof: Since $\overline{A}$ is stable, then for Re(s) > 0

$$n = \text{rank}\begin{bmatrix} Is - \bar{A} \\ C_2 \end{bmatrix} = \text{rank}\begin{bmatrix} Is - A_{11} & -A_{12} \\ 0 & Is - A_{22} \\ I & 0 \end{bmatrix} = \text{rank}\begin{bmatrix} 0 & A_{12} \\ 0 & Is - A_{22} \\ I_{p2} & 0 \end{bmatrix}$$

$$= p_2 + \text{rank}\begin{bmatrix} Is - A_{22} \\ A_{12} \end{bmatrix}. \tag{8-46}$$

This implies that

$$\text{rank}\begin{bmatrix} Is - A_{22} \\ A_{12} \end{bmatrix} = n - p_2 \qquad \text{for } \mathrm{Re}(s) > 0. \tag{8-47}$$

Thus the lemma is proved.

Lemma 8.3 There exists a solution $X = [X_1 \;\; X_2]$ to eq.(8-45).

Proof: Since (A_{12}, A_{22}) is detectable, then there exists a Y and invertible W such that

$$A_{22} + YA_{12} = W^{-1}A_qW = \Lambda \tag{8-48}$$

where Λ is a stable matrix whose eigenvalues consist of the unobservable stable eigenvalues of A_{22} and those we are chosen arbitrarily by selecting a Y. Eq(8-48) actually implies

$$WA_{22} + WYA_{12} = A_qW \tag{8-49}$$

i.e.,

$$X_2 = W \tag{8-50}$$

$$X_1 = WY \tag{8-51}$$

Thus the lemma is proven.

Remark 1. Note that Λ is not unique.

With Lemma 8.2 and 8.3, we are ready to present our main result in the following theorem.

Theorem 8.2 For a given γ, define the parameter matrix

$$Q(s) = \left[\begin{array}{c|c} A_q & B_q \\ \hline C_q & D_q \end{array}\right] \tag{8-52}$$

as

$$A_q = A_{22} + YA_{12} \tag{8-53}$$

$$B_q = YA_{11} + A_{21} - A_qY \tag{8-54}$$

$$C_q = F_2 \tag{8-55}$$

$$D_q = F_1 - F_2 Y \tag{8-56}$$

and

$$F_\infty = [\, F_1 \quad F_2 \,], \tag{8-57}$$

where A_{11}, A_{12}, A_{21} and A_{22} are the partition of $\overline{A}$. If one can choose a matrix Y such that A_q is stable and $\|Q(s)\|_\infty < \gamma$, then there exists an $(n\text{-}p_2)$th order stabilizing controller with the following form

$$K(s) = \left[\begin{array}{c|c} A_q + XZ_\infty B_2 C_q & B_q + X(Z_\infty B_2 D_q - Z_\infty L_\infty) \\ \hline C_q & D_q \end{array}\right], \tag{8-58}$$

where

$$X = [\, Y \quad I \,]. \tag{8-59}$$

Moreover, the H^∞ norm of the closed-loop system is less than γ.

Proof: Choose $W = I$ in the previous lemma, then $A_q = A_{22} + YA_{12}$. Since (A_{12}, A_{22}) is detectable, there always exists a Y such that $A_{22} + YA_{12}$ is stable. Eq.(8-54) is a direct result from eq.(8-44). From eq.(8-29), we have

$$[\, C_q \quad D_q \,] = F_\infty \begin{bmatrix} Y & I \\ I & 0 \end{bmatrix}^{-1} = [\, F_1 \quad F_2 \,] \begin{bmatrix} 0 & I \\ I & -Y \end{bmatrix} = [\, F_2 \quad F_1 - F_2 Y \,].$$

Thus we completed the proof.

Remark 2. Obviously, such a Y is not unique and it is desirable to choose a stabilizing Y such that $\|Q(s)\|_\infty$ is small. However, in case that one can not find a Y such that $\|Q(s)\|_\infty < \gamma$, a larger upper bound on the norm is required.

8.4 Illustrative Example

The generalized plant is given as

$$G(s) = \left[\begin{array}{ccc|ccc:c} -1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & -4 & 1 & 0 & 0 & 2 \\ \hline 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hdashline 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \end{array}\right].$$

By using DGKF formulas and the algorithm we developed in Chapter 5, it is easy to compute the optimal H^∞ norm of the close-loop system, which is 5.31524. A sub-optimal upper bound of the system is chosen as $\gamma_{sub} = 6$. By choosing $Y = [-1\ \ -1]$, we have stable $(A_{22}+YA_{12})$ and $\|Q(s)\|_\infty = 5.54191$ which is less than γ_{sub}. From the formulas given by eq.(8-52) to eq.(8-58) in the previous section, a first order H^∞ controller can be constructed as follows.

$$K(s) = \left[\begin{array}{c|cc} -4.004 & -2.999 & -6.155 \\ \hline -0.259 & -0.277 & -5.535 \end{array}\right] \tag{8-60}$$

With this controller, the eigenvalues and the H^∞ norm of the closed loop system are

$$-4.0,\ -5.691,\ -2.1549,\ -0.9704$$

and 5.541640 respectively.

8.5 Conclusions

In this chapter, a parametrization approach is presented for the reduced-order H^∞ controller design. We showed that there always exists an $(n-p_2)$th order stabilizing controller for an nth order generalized plant with p_2 independent measured outputs. By using the formulas provided, a reduced order H^∞ controller can be easily obtained.

CHAPTER 9

CONTROLLER REDUCTION VIA OBSERVER-BASED CONTROLLER PARAMETRIZATION

Of recent, interest in the conception and development of sophisticated aircraft and spacecrafts has increased. Examples of the spin-off of this interest include such mechanical systems as hypermaneuverable aircrafts and space stations. As these systems get more sophisticated, they become more and more complex. Consequently, conventional modeling techniques and control design strategies become inadequate. This has, by necessity, led to increased research activity in such areas as model reduction [28,31,44,45,46], reduced order controller design [30,36,37,47,48,49], decentralized control [50,51,52], and control/structure interactions [53-56].

This chapter of the report deals with reduced compensator design, which has been tackled from an observer-based compensator point of view. Briefly, this chapter develops two properties related to observer based controller parametrization and pole placement. It shows that the poles of the closed-loop system with the observer-based controller (Q(s)) parametrization are the regulator poles, the observer poles, together with the poles of the added stable parameter matrix K(s). If the controller Q(s) is realized by a minimal realization, the closed loop poles will include all the poles of the added stable parameter matrix K(s) and a subset of the regulator and the observer poles. We parametrize all such K(s) which render Q(s) non-minimal, thereby permitting a minimal realization of Q(s) to serve as a reduced order controller. With such a parametrization available, one could then choose a K(s) in order to best approximate large order controllers, such as the H^∞ compensators, by a lower order controller.

9.1 Observer Based Compensation

One of the most fundamental requirements in control systems design is to make the closed-loop system internally stable. In addition to closed-loop stability, usually the closed-loop system is required to meet some other desired performance criteria. Stabilizing controller parametrization is important because of the following reasons: (1) It provides the full set of the controllers which stabilize the closed-loop system. (2) The full set of

stabilizing controllers is characterized in terms of a stable parameter matrix and the closed-loop system is internally stable if and only if the parameter matrix is stable. (3) The closed-loop transfer function matrix related to the performance can be written as a simple affine function of the parameter matrix and then the control system design problem becomes that of finding a stable parameter matrix such that the closed-loop transfer function matrix meets the desired performance criteria.

The first characterization of the set of all stabilizing controllers in terms of a stable parameter matrix was introduced by Youla et. al. [57] in 1976. Youla's controller characterization was developed based on the fractional factorizations over the ring of polynomial matrices. The only drawback of Youla's characterization is that the stabilizing controller may not be proper. This drawback was removed later by Desoer et. al. [58] in 1980.

Desoer et. al. [58] generalized Youla et. al.'s result based on the fractional factorizations over a general ring. The ring can be chosen as the set of proper stable rational matrices if the given plant is a linear time-invariant system which is represented by a rational matrix. Based on the fractional factorizations over the ring of proper stable rational matrices the set of all proper stabilizing controllers can be characterized in terms of a proper stable parameter matrix. The closed-loop system is internally stable if and only if the parameter matrix is proper stable and the stabilizing controllers are proper if a simple inequality is satisfied.

To use Desoer et. al.'s version of proper stabilizing controller parametrization, it is essential to compute the fractional factorizations over the ring of proper stable rational matrices. Nett et. al. [59] proposed a very convenient state-space method for this computation in 1984. The computation method was developed based on the observer and regulator theories.

Later in 1984, Doyle et. al. [2] showed that the proper stabilizing controller parametrization can be realized as an observer-based controller with an added stable parameter matrix. In 1988 and 1989, Glover and Doyle [8] and Doyle, et. al., [7] offered the two-Riccati-equation approach to solving the standard H^∞ optimization problem. This approach characterizes all possible stabilizing suboptimal H^∞ controllers whose order is not higher than that of the generalized plant. Nonetheless, since the generalized plant model includes the original plant and the models of appropriate weighting functions, this order of the controller is likely to render it non-implementable. Hence, a need for a suitable

methodology for the reduction of this controller arises.

In this chapter we deals with the controller reduction from an observer-based compensator point of view. In essence, the following important properties related to the observer-based controller parametrization and pole placement will be given: (1) The poles of the closed-loop system with the observer-based controller parametrization are the regulator poles, the observer poles, together with the poles of the added stable parameter matrix. (2) If the controller is realized by a minimal realization, the closed-loop poles will include all the poles of the added stable parameter matrix and a subset of the regulator and the observer poles.

In the rest of this section, we will explain the notations to be used in this chapter and briefly review the concept of the stabilizing controller parametrization. We will begin by listing the previous results about the controller parametrizations done by Youla et. al., Desoer et. al., Nett et. al., and Doyle et. al., and present some important properties of the observer-based controller parametrization.

Throughout this chapter, both of the following notations

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \quad \text{and} \quad \{ A, B, C, D \}$$

are used for the same purpose to represent a state-space realization of a system whose transfer function is $C(sI-A)^{-1}B + D$. The sum, A+B, of two sets A with p elements and B with q elements is a set which consists of all elements of A and B. A+B has p+q elements. Assume that B is a subset of A, then the difference, A-B, will consist of all the elements of A except those in B. A-B has p-q elements.

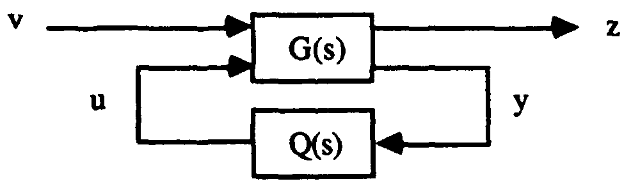


Figure 9.1. Block diagram of a typical control problem.

The concept of stabilizing controller parametrization is briefly described as follows. Consider the block diagram in Fig. 9.1 where v is the exogenous input vector which may consist of the disturbances, noises, and the commands, u is the control input vector through which the behavior of the system can be modified, z is the controlled output vector

which is composed of all the variables to be controlled, and y is the measured output vector which consists of all the measurable quantities available for feedback. The plant G(s) is given by

$$G(s) = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] \tag{9-1}$$

The objective of a typical control problem is to find a proper controller Q(s) which stabilizes the closed-loop system and the H^∞ (or H^2) norm of the closed-loop transfer function matrix $\Phi(s)$ from v to z is minimized. The first step to solve the problem is to find the set of all proper controllers which make the closed-loop system internally stable. Then in the set of all proper stabilizing controllers, one will be chosen such that the H^∞ (or H^2) norm of $\Phi(s)$ is minimized.

The stabilizing controller parametrization we are interested in has the following two properties: (1) All the proper stabilizing controllers can be characterized in terms of a proper stable parameter matrix K(s) and the closed-loop system is internally stable if and only if K(s) is stable. (2) The transfer function matrix $\Phi(s)$ is a simple affine function of the parameter matrix K(s).

After the stabilizing controller parametrization, the above control problem becomes that of finding a proper stable matrix K(s) such that $\|\Phi\|_\infty$ (or $\|\Phi\|_2$) is minimized. Property (2) of the last paragraph is important since it will make the H^∞ (or H^2) optimization problem easy to solve.

9.1.1 Preliminaries

The previous results related to the controller parametrization will be briefly reviewed in this section. The following theorem was originally developed by Youla et. al. [57] and later modified by Desoer et. al. [58].

Theorem 9.1: *[57,58] (Youla's Controller Parametrization)*
Consider the system in Fig. 9.1. Assume that the realization in (9.1) is minimal and the subsystem $G_{22}(s)$ is stabilizable and detectable. Let $M_2(s)$, $N_2(s)$, $X_2(s)$, $Y_2(s)$, $M_1(s)$, $N_1(s)$, $X_1(s)$, and $Y_1(s)$ be proper stable rational matrices such that

$$\begin{bmatrix} M_2(s) & N_2(s) \\ -Y_1(s) & X_1(s) \end{bmatrix} \begin{bmatrix} X_2(s) & -N_1(s) \\ Y_2(s) & M_1(s) \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \tag{9-2}$$

and

$$M_2(s)^{-1} N_2(s) = G_{22}(s) \tag{9-3}$$

Then the set of all proper stabilizing controllers can be described as

$$\Big\{ Q(s) \;|\; Q(s) = [\, M_1(s)K(s) + Y_2(s) \,]\,[\, N_1(s)K(s) - X_2(s) \,]^{-1}$$
$$\text{with } K(s) \text{ proper stable and } |N_1(\infty)K(\infty) - X_2(\infty)| \neq 0 \Big\} \tag{9-4}$$

and the closed-loop transfer function matrix $\Phi(s)$ from v to z is an affine function of the parameter matrix $K(s)$,

$$\Phi(s) = G_{11}(s) - G_{12}(s)Y_2(s)M_2(s)G_{21}(s) - G_{12}(s)M_1(s)\,K(s)\,M_2(s)G_{21}(s) \tag{9-5}$$

To use Theorem 9.1, we need to construct the proper stable rational matrices in (9-2) and (9-3). Nett et. al. [59] proposed a convenient state-space approach for this construction. That is, the following realizations

$$\begin{bmatrix} M_2(s) & N_2(s) \\ -Y_1(s) & X_1(s) \end{bmatrix} = \left[\begin{array}{c|cc} A+HC_2 & H & B_2+HD_{22} \\ \hline C_2 & I & D_{22} \\ -F & 0 & I \end{array}\right] \tag{9-6a}$$

and

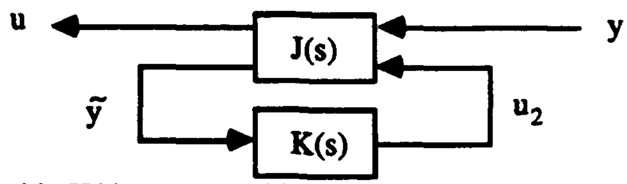
$$\begin{bmatrix} X_2(s) & -N_1(s) \\ Y_2(s) & M_1(s) \end{bmatrix} = \left[\begin{array}{c|cc} A+B_2F & H & B_2 \\ \hline -(C_2+D_{22}F) & I & -D_{22} \\ F & 0 & I \end{array}\right] \tag{9-6b}$$

are proper stable and satisfy (9-2) and (9-3) where F and H can be arbitrarily chosen such that $A+B_2F$ and $A+HC_2$ are stable.

Doyle et. al. [2] showed that if (9-6a) and (9-6b) are used to realize the proper stable rational matrices in (9-2) and (9-3) and let

$$J(s) = \begin{bmatrix} J_{11}(s) & J_{12}(s) \\ J_{21}(s) & J_{22}(s) \end{bmatrix} = \left[\begin{array}{c|cc} A+B_2F+HC_2+HD_{22}F & -H & -(B_2+HD_{22}) \\ \hline F & 0 & -I \\ -(C_2+D_{22}F) & I & D_{22} \end{array}\right] \qquad (9\text{-}7)$$

then the set of proper stabilizing controllers described in Theorem 9.1 will have a structure as that shown in Fig. 9.2.



with K(s) proper stable and $I - D_{22}K(\infty)$ invertible.
Figure 9.2. Structure of stabilizing controller parametrization.

Replace the controller Q(s) in Fig. 9.1 by the structure of Fig. 9.2, then the closed-loop system can be redrawn as that shown in Fig. 9.3.

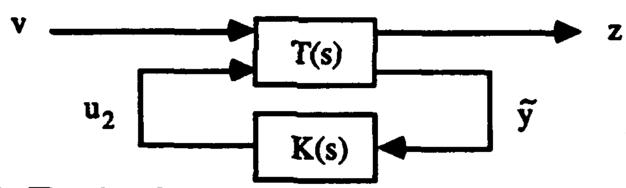


Figure 9.3. The closed-loop system in terms of a parameter matrix K(s).

In Fig.9.3, the open-loop transfer function matrix from u_2 to $\tilde{y}$, $T_{22}(s)$, is zero. Therefore, the closed-loop transfer function matrix from v to z, i.e., $\Phi(s)$, is a simple affine function of the parameter matrix K(s). That is,

$$\Phi(s) = T_{11}(s) + T_{12}(s)\,K(s)\,T_{21}(s) \qquad (9\text{-}8)$$

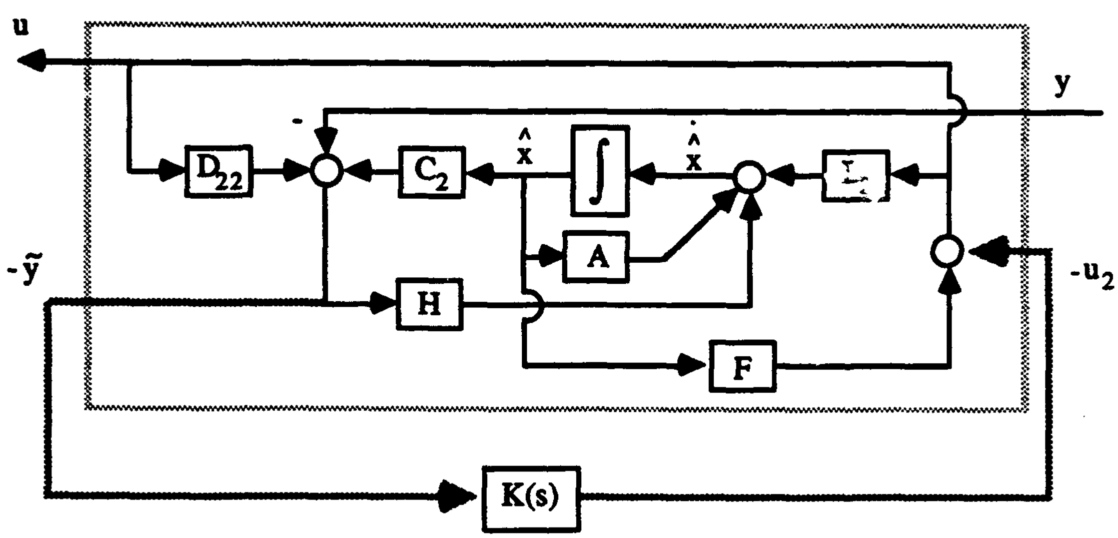
where the realizations of $T_{11}(s)$, $T_{12}(s)$, $T_{21}(s)$ are given by

$$T_{11}(s) = \left[\begin{array}{cc|c} A+B_2F & -B_2F & B_1 \\ 0 & A+HC_2 & B_1+HD_{21} \\ \hline C_1+D_{12}F & -D_{12}F & D_{11} \end{array}\right] \qquad (9\text{-}9a)$$

$$T_{12}(s) = \left[\begin{array}{c|c} A+B_2F & B_2 \\ \hline C_1+D_{12}F & D_{12} \end{array}\right] \qquad (9\text{-}9b)$$

$$T_{21}(s) = \left[\begin{array}{c|c} A+HC_2 & B_1+HD_{21} \\ \hline C_2 & D_{21} \end{array}\right] \qquad (9\text{-}9c)$$

Doyle et. al. [2] also pointed out that the structure of the stabilizing controller parametrization in Fig. 9.2 can be realized as an observer-based controller with an added stable dynamics K(s). The realization is shown in Fig. 9.4.



with K(s) proper stable and $I - D_{22}K(\infty)$ invertible
Figure 9.4. The observer-based controller parametrization.

Note that in Fig. 9.4 the block diagram inside the dotted-line box is the well-known full-order observer-based controller [17].

9.1.2 Main results

In Fig. 9.1, the internal stability of the closed-loop system depends only on $G_{22}(s)$ and Q(s), i.e., the interconnected system shown in Fig. 9.5.

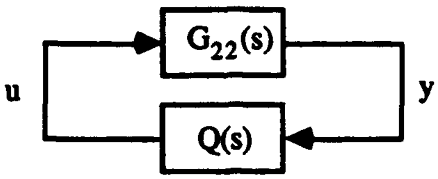


Figure 9.5. Equivalent system to Fig. 9.2 for internal stability.

In this section, the controller Q(s) in Fig. 9.5 is replaced by the block diagram of Fig. 9.4 which is the observer-based controller with an added dynamics K(s).

In the following theorem we will show that the poles of the closed-loop system with the observer-based controller parametrization in Fig. 9.4 are the regulator poles (the eigenvalues of $A+B_2F$), the observer poles (the eigenvalues of $A+HC_2$), together with the poles of the parameter matrix K(s). In the design of the observer-based controller, F and H are chosen such that the eigenvalues of $A+B_2F$ and $A+HC_2$ are stable. Therefore, the closed-loop system is internally stable if and only if the parameter matrix K(s) is proper stable. The proof is quite straightforward and is done completely in the state space without referring to the derivations used by Youla et. al., Desoer et. al., and Doyle et. al..

Theorem 9.2: *(Observer-based Controller Parametrization)*

Consider the closed-loop system in Fig. 9.5. Assume that $G_{22}(s) = \{A, B_2, C_2, D_{22}\}$ with order n is stabilizable and detectable and the controller Q(s) is replaced by the observer-based controller with an added m-th order dynamics K(s) as shown in Fig. 9.4. Then the set of the closed-loop poles is composed of the n eigenvalues of $A+B_2F$, the n eigenvalues of $A+HC_2$, and the m poles of the added dynamics K(s). That is, the set of the closed-loop poles is

$$P_{\text{closed-loop}} = P_{\text{regulator}} + P_{\text{observer}} + P_{K(s)} \tag{9-10a}$$

where

$$P_{\text{regulator}} = \{ \text{ n eigenvalues of } A+B_2F \} \tag{9-10b}$$

$$P_{\text{observer}} = \{ \text{ n eigenvalues of } A+HC_2 \} \tag{9-10c}$$

and

$$P_{K(s)} = \{ \text{ m poles of } K(s) \} \tag{9-10d}$$

<u>Proof</u>: The dynamic equations of $G_{22}(s)$ is given by

$$\dot{x} = A x + B_2 u \tag{9-11a}$$

$$y = C_2 x + D_{22} u \tag{9-11b}$$

The dynamic equations of the observer-based controller in Fig. 9.4, i.e., the block diagram inside the dotted-line box, can be written as follows,

$$\dot{\hat{x}} = (A+B_2F+HC_2+HD_{22}F)\,\hat{x} + \begin{bmatrix} -H & -(B_2+HD_{22}) \end{bmatrix} \begin{bmatrix} y \\ u_2 \end{bmatrix} \tag{9-12a}$$

$$\begin{bmatrix} u \\ \tilde{y} \end{bmatrix} = \begin{bmatrix} F \\ -(C_2+D_{22}F) \end{bmatrix} \hat{x} + \begin{bmatrix} 0 & -I \\ I & D_{22} \end{bmatrix} \begin{bmatrix} y \\ u_2 \end{bmatrix} \tag{9-12b}$$

Assume that the added dynamics K(s) is described by the following <u>minimal</u> realization

$$\dot{k} = \tilde{A}\,k + \tilde{B}\,\tilde{y} \tag{9-13a}$$

$$u_2 = \tilde{C}\,k + \tilde{D}\,\tilde{y} \tag{9-13b}$$

The controller Q(s) is just a combination of (9-12) and (9-13). From (9-12) and (9-13), we have the dynamic equations of the controller Q(s) as follows,

$$\begin{bmatrix} \dot{\hat{x}} \\ \dot{k} \end{bmatrix} = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} \begin{bmatrix} \hat{x} \\ k \end{bmatrix} + \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} y \tag{9-14a}$$

$$u = \begin{bmatrix} \gamma_1 & \gamma_2 \end{bmatrix} \begin{bmatrix} \hat{x} \\ k \end{bmatrix} + \delta\, y \tag{9-14b}$$

where

$$\beta_1 = -H - (B_2+HD_{22})\,(I-\tilde{D}D_{22})^{-1}\tilde{D} \tag{9-15a}$$

$$\beta_2 = \tilde{B} + \tilde{B}\,D_{22}\,(I-\tilde{D}D_{22})^{-1}\tilde{D} \tag{9-15b}$$

$$\gamma_1 = F + (I-\tilde{D}D_{22})^{-1}\tilde{D}\,(C_2+D_{22}F) \tag{9-15c}$$

$$\gamma_2 = -(I-\tilde{D}D_{22})^{-1}\tilde{C} \tag{9-15d}$$

$$\alpha_{11} = A + HC_2 + (B_2+HD_{22})\,\gamma_1 \tag{9-15e}$$

$$= A + B_2F - \beta_1\,(C_2+D_{22}F) \tag{9-15f}$$

$$\alpha_{12} = (B_2+HD_{22})\,\gamma_2 \tag{9-15g}$$
$$\alpha_{21} = -\beta_2\,(C_2+D_{22}F) \tag{9-15h}$$
$$\alpha_{22} = \tilde{A} - \tilde{B}\,D_{22}\,\gamma_2 \tag{9-15i}$$
$$\delta = -(I-\tilde{D}D_{22})^{-1}\tilde{D} \tag{9-15j}$$

Now, combine the dynamic equations of the controller Q(s) and the plant $G_{22}(s)$, i.e., equations (9-14) and (9-11). Then the state equation of the closed-loop system can be obtained as follows

$$\begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \\ \dot{k} \end{bmatrix} = \begin{bmatrix} \#_{11} & \#_{12} & \#_{13} \\ \#_{21} & \#_{22} & \#_{23} \\ \#_{31} & \#_{32} & \#_{33} \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \\ k \end{bmatrix} \tag{9-16}$$

where

$$\#_{11} = A + B_2\,\delta\,(I - D_{22}\delta)^{-1}C_2 \tag{9-17a}$$
$$\#_{12} = B_2\gamma_1 + B_2\,\delta\,(I - D_{22}\delta)^{-1}D_{22}\gamma_1 \tag{9-17b}$$
$$\#_{13} = B_2\gamma_2 + B_2\,\delta\,(I - D_{22}\delta)^{-1}D_{22}\gamma_2 \tag{9-17c}$$
$$\#_{21} = \beta_1\,(I - D_{22}\delta)^{-1}C_2 \tag{9-17d}$$
$$\#_{22} = \alpha_{11} + \beta_1\,(I - D_{22}\delta)^{-1}D_{22}\gamma_1 \tag{9-17e}$$
$$\#_{23} = \alpha_{12} + \beta_1\,(I - D_{22}\delta)^{-1}D_{22}\gamma_2 \tag{9-17f}$$
$$\#_{31} = \beta_2\,(I - D_{22}\delta)^{-1}C_2 \tag{9-17g}$$
$$\#_{32} = \alpha_{21} + \beta_2\,(I - D_{22}\delta)^{-1}D_{22}\gamma_1 \tag{9-17h}$$
$$\#_{33} = \alpha_{22} + \beta_2\,(I - D_{22}\delta)^{-1}D_{22}\gamma_2 \tag{9-17i}$$

Additionally, we make the following observations, which can be easily shown by directly manipulating the equations (9-17) and (9-15).

<u>Observation 1</u>: $\#_{32} = -\#_{31}$ (9-18a)

<u>Observation 2</u>: $\#_{11} + \#_{12} = \#_{21} + \#_{22} = A + B_2F$ (9-18b)

<u>Observation 3</u>: $\#_{23} = \#_{13}$ (9-18c)

<u>Observation 4</u>: $-\#_{12} + \#_{22} = A + HC_2$ (9-18d)

<u>Observation 5</u>: $\#_{33} = \tilde{A}$ (9-18e)

Define the observer reconstruction error

$$\tilde{x} = x - \hat{x} \tag{9-19}$$

then we have

$$\begin{bmatrix} \dot{x} \\ \dot{\tilde{x}} \\ \dot{k} \end{bmatrix} = \begin{bmatrix} \#_{11}+\#_{12} & -\#_{12} & \#_{13} \\ \#_{11}+\#_{12}-\#_{21}-\#_{22} & -\#_{12}+\#_{22} & \#_{13}-\#_{23} \\ \#_{31}+\#_{32} & -\#_{32} & \#_{33} \end{bmatrix} \begin{bmatrix} x \\ \tilde{x} \\ k \end{bmatrix}$$

$$= \begin{bmatrix} A+B_2F & -\#_{12} & \#_{13} \\ 0 & A+HC_2 & 0 \\ 0 & -\#_{32} & \tilde{A} \end{bmatrix} \begin{bmatrix} x \\ \tilde{x} \\ k \end{bmatrix} \tag{9-20}$$

which is related to (9-16) by the similarity transformation

$$\begin{bmatrix} I & 0 & 0 \\ I & -I & 0 \\ 0 & 0 & I \end{bmatrix} \tag{9-21}$$

Therefore, from the structure of the matrix in (9-20), the characteristic values of the closed-loop system are those of $A+B_2F$, $A+HC_2$, and $\tilde{A}$. This completes the proof of Theorem 9.2.

It is well known that in the observer-based controller design the closed-loop poles are the regulator poles (the eigenvalues of $A+B_2F$) and the observer poles (the eigenvalues of $A+HC_2$) [17]. In Theorem 9.2 we just showed that the above property still remains when we add a dynamics K(s) to the observer-based controller as shown in Fig. 9.4. The eigenvalues of $A+B_2F$ and $A+HC_2$ are still parts of the closed-loop poles after we add K(s) to the controller. Adding K(s) only introduces additional poles to the closed-loop system and the added closed-loop poles are the poles of K(s). If F and H have been chosen such

that $A+B_2F$ and $A+HC_2$ are stable, then the closed-loop system with the observer-based controller parametrization will be internally stable if and only if the parameter matrix K(s) is proper stable. From Fig. 9.4, it is easy to see that the controller Q(s) is proper if K(s) is proper and $I - D_{22}K(\infty)$ is invertible.

With the observer-based controller parametrization, the closed-loop transfer function matrix from v to z, i.e., $\Phi(s)$, is a simple affine function of the parameter matrix K(s). That is,

$$\Phi(s) = T_{11}(s) + T_{12}(s)\,K(s)\,T_{21}(s) \tag{9-22}$$

where $T_{11}(s)$, $T_{12}(s)$, and $T_{21}(s)$ are given by (9-9). The added dynamics K(s) is a proper stable rational matrix to be chosen such that $I-D_{22}K(\infty)$ is invertible and $\Phi(s)$ has some desired performance. No matter which K(s) is to be selected, we always have clear idea that the closed-loop poles will be the eigenvalues of $A+B_2F$ and $A+HC_2$ together with the poles of the added dynamics K(s) if the controller is realized as that shown in Fig. 9.4.

Assume that the orders of the plant and the parameter matrix K(s) are n and m respectively. If the controller is realized as that shown in Fig. 9.4, then the order of the controller is n+m and the closed-loop system has 2n+m poles described by the set $P_{closed\text{-}loop}$ in (9-10). The realization of the controller in Fig. 9.4 may not be minimal. Suppose it is not and there are r poles in the controller either uncontrollable or unobservable, then the controller can be realized by a minimal realization with order n+m-r and the number of closed-loop poles will be reduced to 2n+m-r.

In the following theorem, we will show that the uncontrollable or unobservable poles of the controller of Fig. 9.4 must be the eigenvalues of $A+B_2F$ or $A+HC_2$ and the closed-loop poles will always include all m poles of the added stable dynamics K(s) no matter the realization of the controller is minimal or not. If r pole-zero cancellations occur in the controller, then the closed-loop poles will include m poles of K(s), and 2n-r eigenvalues out of the set $P_{regulator} + P_{observer}$ which was defined in (9-10).

Theorem 9.3: *Consider the closed-loop system in Fig. 9.5. Assume that $G_{22}(s) = \{A, B_2, C_2, D_{22}\}$ with order n is stabilizable and detectable and the controller Q(s) is replaced by the observer-based controller with an added m-th order dynamics K(s) as shown in Fig. 9.4. A minimal realization of K(s) is given by (9-13). Define $P_{regulator}$, $P_{observer}$ and $P_{K(s)}$ by (9-10b), (9-10c), and (9-10d) respectively and let*

$$P_{removal} = \{ \text{ the controller poles which are either uncontrollable or unobservable} \} \quad (9\text{-}23)$$

Then

$$P_{regulator} + P_{observer} \supset P_{removal} \quad (9\text{-}24)$$

and the closed-loop system with the minimal order controller will have a set of poles described by the following

$$P_{closed\text{-}loop \text{ with min. controller}} = P_{K(s)} + (P_{regulator} + P_{observer} - P_{removal}) \quad (9\text{-}25)$$

Proof: Let $N_G(s)D_G(s)^{-1}$ with deg $|D_G(s)| = n$ and $D_Q(s)^{-1}N_Q(s)$ with deg $|D_Q(s)| = n+m$ be a right MFD (matrix fraction description) of $G_{22}(s)$ and a left MFD of $Q(s)$ respectively [19]. It is well known that the characteristic polynomial of the closed-loop system is

$$\phi_{closed\text{-}loop}(s) = | D_Q(s)D_G(s) - N_Q(s)N_G(s) | \quad (9\text{-}26)$$

Let $L_Q(s)$ be a greatest common left divisor of $D_Q(s)$ and $N_Q(s)$. That is,

$$D_Q(s) = L_Q(s)\, \hat{D}_Q(s) \;, \quad N_Q(s) = L_Q(s)\, \hat{N}_Q(s) \quad (9\text{-}27)$$

where $\hat{D}_Q(s)$ and $\hat{N}_Q(s)$ are left coprime. It is easy to see that the zeros of $L_Q(s)$ are the uncontrollable or the unobservable poles of the controller realization in Fig. 9.4. That is,

$$\{ \text{ zeros of } L_Q(s) \} = P_{removal} \quad (9\text{-}28)$$

Plug (9-27) into (9-26), we have

$$\phi_{closed\text{-}loop}(s) = | L_Q(s) | \cdot | \hat{D}_Q(s)\, D_G(s) - \hat{N}_Q(s)\, N_G(s) | \quad (9\text{-}29)$$

The zeros of $| \hat{D}_Q(s)\, D_G(s) - \hat{N}_Q(s)\, N_G(s) |$ are the poles of the closed-loop system with a minimal controller realization. From Theorem 9.2, (9-28) and (9-29), we can see that

$$P_{closed\text{-}loop \text{ with min. controller}} + P_{removal} = P_{regulator} + P_{observer} + P_{K(s)} \quad (9\text{-}30)$$

To complete the proof, we need to show that $P_{removal}$ is a subset of $P_{regulator}$ + $P_{observer}$ when K(s) is minimal. First, assume that the state-space representation (9-14) of

the controller Q(s) is unobservable. Then by PBH test [19], there exists a nonzero vector ξ such that

$$\begin{bmatrix} \gamma_1 & \gamma_2 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = 0; \qquad \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \xi \tag{9-31a}$$

$$\begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \lambda \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \tag{9-31b}$$

for some eigenvalue λ of (9-14). Note that it is the eigenvalue λ that is unobservable. From (9-31b), we get

$$\alpha_{11}\xi_1 + \alpha_{12}\xi_2 = \lambda \xi_1 \tag{9-32a}$$

which, by using (9-15e) and (9-15g), is rewritten as

$$(A + HC_2)\xi_1 + (B_2 + HD_{22})(\gamma_1\xi_1 + \gamma_2\xi_2) = \lambda \xi_1 \tag{9-32b}$$

In view of (9-31a), the above equation reduces to

$$(A + HC_2)\xi_1 = \lambda \xi_1 \tag{9-32b}$$

which clearly establishes that the unobservable eigenvalue belongs to $P_{observer}$.

Proceeding similarly, it can be shown that if (9-14) is uncontrollable then the uncontrollable eigenvalue belongs to $P_{regulator}$.

Note in the above development that $\xi_1 = 0$ contradicts the minimality assumption of K(s). Thus,

$$P_{observer} \supset P_{unobservable} \tag{9-33a}$$

$$P_{regulator} \supset P_{uncontrollable} \tag{9-33b}$$

where $P_{unobservable}$ is the set of all the unobservable poles of the controller Q(s). $P_{uncontrollable}$ is defined similarly. This completes the proof of Theorem 9.3.

The results of the current section can be summarized as follows. The poles of the

closed-loop system with the observer-based controller parametrization shown in Fig. 9.4 can be classified into three groups and each group of poles can be independently determined. These three groups of poles are the regulator poles (the eigenvalues of $A+B_2F$), the observer poles (the eigenvalues of $A+HC_2$), and the poles of the added dynamics K(s). F, H, and K(s) are free parameters to be chosen such that the closed-loop transfer function matrix $\Phi(s)$ has some optimal performance subject to the following constraints: $A+B_2F$ and $A+HC_2$ are stable and K(s) is proper stable with $I-D_{22}K(\infty)$ invertible.

If the realization of the controller in Fig. 9.4 is not minimal, then the uncontrollable and/or unobservable controller poles can be removed and the order of the controller is minimized. The set of these removable controller poles is a subset of the regulator and the observer poles. The poles of the closed-loop system with the minimal order controller will include all the poles of the parameter matrix K(s) and some of the regulator and the observer poles which are not the removable controller poles.

9.2 Controller Reduction

Motivated by the results of the previous section, we pose the controller reduction problem as follows: given the plant P(s) as

$$\begin{aligned} \dot{x}_p &= A_p x_p + B_p u \quad ; \quad x_p \in R^{n_p} ; \ u \in R^m \\ z_p &= C_p x_p + D_p u \quad ; \quad z_p \in R^l \end{aligned} \tag{9-34}$$

and the observer-based controller J(s) as

$$\begin{aligned} \dot{x}_j &= A_p x_j + B_p u + B_j(z_p - z_j) \quad ; \quad x_j \in R^{n_p} \\ z_j &= C_p x_j + D_p u \\ u &= u_j \\ u_j &= C_j x_j \end{aligned} \tag{9-35}$$

obtain a reduced order controller G(s) whose order n_g is required to be less than that of J(s):

$$\begin{aligned} \dot{x}_g &= A_g x_g + B_g z_p \quad ; \quad x_g \in R^{n_g} \\ u &= C_g x_g + D_g z_p \end{aligned} \tag{9-36}$$

Clearly, if the above realization for J(s) in non-minimal, one could obtain its minimal realization and use that as the required compensator G(s). Several authors have

been motivated by this approach; for example see references [32,36,37].

We take a different approach, which is motivated from the observer-based controller parametrization of the previous section. Briefly, our approach adopts these steps:

1. Consider the control u as consisting of two components

$$u = u_j + u_k \tag{9-37}$$

2. Let u_k be generated from an additional dynamic parameter K(s) inserted between the measurement residual $(z_p - z_j)$ and the control u_k as given below.

$$\begin{aligned} \dot{x}_k &= A_k x_k + B_k(z_p - z_j) \ ; \ x_k \in R^{n_k} \\ u_k &= C_k x_k + D_k(z_p - z_j) \end{aligned} \tag{9-38}$$

   where the parameters $\{A_k, B_k, C_k, D_k\}$ are yet to be selected - see Step 4 below. Note that we are increasing the order of the overall controller by appending the additional dynamics K(s).

3. Now, the overall controller Q(s), consisting of both the given controller J(s) and the additional dynamics K(s), can be realized as

$$\begin{aligned} \dot{x}_q &= A_q x_q + B_q z_p \ ; \ x_q \in R^{n_q} \ ; \ n_q = n_p + n_k \\ u &= C_q x_q + D_q z_p \end{aligned} \tag{9-39}$$

4. Choose the parameters $\{A_k, B_k, C_k, D_k\}$ so that Q(s) is non-minimal. Obtain the reduced controller G(s) as the minimal realization of Q(s).

The purpose of this study is to show that the state-space parameters $\{A_k, B_k, C_k, D_k\}$ of K(s) can be selected to render Q(s) non-minimal, and to render the minimal order of Q(s) to be less than n_p.

We first rewrite a few properties of observer-based controllers from last section in the notation of the current section.

Theorem 9.2 : The eigenvalues $\Lambda_{c/l}$ of the overall closed loop system are given by

$$\Lambda_{c/l} = \Lambda_r \dot{\cup} \Lambda_o \dot{\cup} \Lambda_k \tag{9-40}$$

where

Λ_r	=	$\Lambda(A_p+B_pC_j)$	=	eigenvalues of the regulator,
Λ_o	=	$\Lambda(A_p-B_jC_p)$	=	eigenvalues of the observer, and

$\Lambda_k \quad = \quad \Lambda(A_k) \quad = \quad$ eigenvalues of the parameter K(s).

Notice that the addition of K(s) does not change the eigenvalues of the closed loop system; it only adds additional eigenvalues. Now, if the controller Q(s) is non-minimal then there are some eigenvalues which are either unobservable and/or uncontrollable, and these eigenvalues can be removed from Q(s) to obtain a smaller (minimal) order controller G(s). When the state space realization of K(s) is assumed to be minimal, the following result ensues.

Theorem 9.3 : Let Λ_{uo} and Λ_{uc} be the unobservable and uncontrollable eigenvalues of Q(s), respectively. Then

$$\Lambda_{uo} \subset \Lambda_o \ \ \& \ \ \Lambda_{uc} \subset \Lambda_r \tag{9-41}$$

Remark: The $(2n_p - n_{uo} - n_{uc} + n_k)$ eigenvalues of the reduced closed loop system will be the remaining eigenvalues of the regulator and the observer together with the eigenvalues of K(s). Since the regulator and the observer are normally designed to be stable, the stability of the reduced closed loop system is therefore guaranteed.

Based on these theorems, we present the following main theorem, whose proof is outlined in the Appendix.

Theorem 9.4: *Let Λ_{uo} and Λ_{uc} be some eigenvalues of $Q(s)$ that are to be made unobservable and uncontrollable, respectively. Then the parameters of $K(s)$ to achieve this must satisfy*

$$\begin{bmatrix} A_k & B_k \\ C_k & D_k \end{bmatrix} \begin{bmatrix} \Psi_k \\ -C_p\Psi_o \end{bmatrix} = \begin{bmatrix} \Psi_k\Lambda_{uo} \\ -C_j\Psi_o \end{bmatrix} \tag{9-42a}$$

$$\begin{bmatrix} \Phi_k & \Phi_r B_p \end{bmatrix} \begin{bmatrix} A_k & B_k \\ C_k & D_k \end{bmatrix} = \begin{bmatrix} \Lambda_{uc}\Phi_k & -\Phi_r B_j \end{bmatrix} \tag{9-42b}$$

for some Ψ_k and Φ_k , and where Ψ_o and Φ_r satisfy

$$\Phi_r(A_p+B_pC_j) = \Lambda_{uc}\Phi_r \tag{9-43a}$$

$$(A_p-B_jC_p)\Psi_o = \Psi_o\Lambda_{uo} \tag{9-43b}$$

The order n_g of the minimal realization of Q(s) is then given by

$$n_g = n_p - n_{uo} - n_{uc} + n_k . \tag{9-44}$$

It is possible to select K(s) such that n_g is less than n_p, as shown in the example below.

In what follows we study a special case of this problem when K(s) is static (n_k = 0), i.e., the constant matrix D_k is the only parameter that describes K(s). The order of the reduced controller would then be $n_g = n_p - n_{uo} - n_{uc}$. Results presented below are easily extended for the more general case of dynamic K(s).

Corollary 9.1: *The necessary and sufficient condition for the parameter D_k (i.e. static K(s)) to achieve this is the satisfaction of the following equations:*

$$D_k C_p \Psi_o = C_j \Psi_o, \text{ and } \Phi_r B_p D_k = -\Phi_r B_j, \tag{9-45}$$

subject to (9-43a) and (9-43b).

Though not necessary, Φ_r and Ψ_o can be seen as consisting of some left eigenvectors of the regulator and some right eigenvectors of the observer, respectively. Note that if either n_{uo} or n_{uc} is equal to zero then only one of the above equations need to be satisfied. For instance, suppose that the given controller J(s) is not completely observable, which may happen as in the case of LQG controllers. Then there exists a Ψ_o such that $C_j\Psi_o = 0$, and hence a $D_k = 0$ will remove these unobservable eigenvalues. This result, along with those from the previous section, establishes the following:

Corollary 9.2: *If a given controller J(s) is non-minimal, then the closed loop system using a minimal realization of J(s) as its reduced controller would contain only the observable eigenvalues of the observer and the controllable eigenvalues of the regulator.*

The above results are pertaining only to the stability of the closed loop system. An equally important issue in controller reduction is the performance of the closed loop system. This issue can be addressed, at least in a suboptimal sense, if one could determine the eigenvalues that are least significant in the performance. The Modal Cost Analysis [60] offers a suitable methodology for identifying such eigenvalues. However, only those eigenvalues that allow the satisfaction of the above equations can be removed. The following result assists in identifying such removable eigenvalues.

Theorem 9.5:

(a) When $n_{uc} = 0$ and $n_{uo} > 0$, an order-reducing D_k exists if and only if

$$\mathrm{Kernel}(C_p\Psi_o) \subseteq \mathrm{Kernel}(C_j\Psi_o), \tag{9-46a}$$

with the general solution

$$D_k = C_j\Psi_o(C_p\Psi_o)^- + Z\,[\,I-(C_p\Psi_o)(C_p\Psi_o)^-\,]. \tag{9-46b}$$

(b) When $n_{uo} = 0$ and $n_{uc} > 0$, an order-reducing D_k exists if and only if

$$\mathrm{Image}(\Phi_r B_j) \subseteq \mathrm{Image}(\Phi_r B_p), \tag{9-47a}$$

with the general solution

$$D_k = -(\Phi_r B_p)^-\Phi_r B_j + [\,I-(\Phi_r B_p)^-(\Phi_r B_p)\,]\,Z\,. \tag{9-47b}$$

(c) Under the assumption of $\Lambda_{uo} \cap \Lambda_{uc} = \varnothing$, when both $n_{uo} > 0$ and $n_{uc} > 0$, a D_k satisfying the equations of Corollary 9.1 exists if and only if

$$\Phi_r\Psi_o = 0, \tag{9-48a}$$

along with the satisfaction of (9-46a) and (9-47a), with the general solution

$$\begin{aligned} D_k = &-(\Phi_r B_p)^-\Phi_r B_j + C_j\Psi_o(C_p\Psi_o)^- - (\Phi_r B_p)^-\Phi_r B_p C_j\Psi_o(C_p\Psi_o)^- \\ &+ [\,I-(\Phi_r B_p)^-(\Phi_r B_p)\,]\,Z\,[\,I-(C_p\Psi_o)(C_p\Psi_o)^-\,] \end{aligned} \tag{9-48b}$$

where $(\bullet)^-$ denotes generalized inverse, and Z is arbitrary.

Under the assumption of $\Lambda_{uo} \cap \Lambda_{uc} = \varnothing$,the above theorem suggests this procedure for determining the removable eigenvalues: Compute the complete set of the left eigenvectors of the regulator, and the right eigenvectors of the observer; denote the first set by Φ and the second by Ψ; use the zero elements of the product $[\Phi\Psi]$ to identify the appropriate n_{uc} left and n_{uo} right eigenvectors; collect these left eigenvectors to form Φ_r and the right eigenvectors to form Ψ_o. Then, if the conditions (9-46a) and (9-47a) are also satisfied, the feed-through loop determined according to Part (c) of Theorem 9.5 will yield a reduced controller of order $(n_p-n_{uo}-n_{uc})$. The absence of any zero elements in $[\Phi\Psi]$ implies that a feed-through loop does not exist which would simultaneously render some observer eigenvalues unobservable and some regulator eigenvalues uncontrollable. (A dynamic K(s) may however be constructed as shown later in this report). It may, however, be possible to remove some eigenvalues of only the observer or only the controller. This is because Part (c) of the above theorem is pertaining to the simultaneous satisfaction of the two equations of Corollary 9.1, which is not necessary when either n_{uo} or n_{uc} is equal to

zero. In this case we have Parts (a) and (b) of the theorem.

An Example:

Let the plant parameters be

$$A_p = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \;;\; B_p = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \;;\; C_p = \begin{bmatrix} 1 & 1 \end{bmatrix} \;;\; D_p = 0,$$

and let the regulator gain C_j and the observer gain B_j be

$$C_j = \begin{bmatrix} -4 & -3 \end{bmatrix} \;;\; B_j = \begin{bmatrix} 4 \\ 0 \end{bmatrix}.$$

The order of the controller J(s) is 2, and the eigenvalues of the closed loop system consisting of P(s) and J(s) are those of the regulator Λ_r and the observer Λ_o where, for this example,

$$\Lambda_r = \{ -1, -3 \} \text{ and } \Lambda_o = \{ -2, -2 \}.$$

Suppose that we wish to reduce the order of the controller by removing an -2 eigenvalue of the observer and the -1 eigenvalue of the regulator. This can be achieved by introducing K(s) whose parameters are

$$A_k = A_k \;;\; B_k = A_k + 2 \;;\; C_k = A_k + 1 \;;\; D_k = A_k + 6,$$

for any arbitrary, but stable A_k. The corresponding realization of Q(s) has the parameters

$$A_q = \begin{bmatrix} A_k-2 & A_k-1 & -A_k-1 \\ 1 & 0 & 0 \\ -A_k-2 & -A_k-2 & A_k \end{bmatrix} ; \; B_q = \begin{bmatrix} -A_k-2 \\ 0 \\ A_k+2 \end{bmatrix} ;$$

$$C_q = \begin{bmatrix} A_k+2 & A_k+3 & -A_k-1 \end{bmatrix} ; \; D_q = \begin{bmatrix} -A_k-6 \end{bmatrix}$$

which has an unobservable eigenvalue of -2 and an uncontrolable eigenvalue of -1, as required. The reduced order controller G(s), which is the minimal realization of Q(s), is then given by the parameters

$$A_g = 2A_k+1 \;;\; B_g = -(A_k+2) \;;\; C_g = 2A_k+3$$

$D_g = -(A_k+6)$

By Theorem 9.3 above, the eigenvalues of the closed loop system consisting of P(s) and the reduced controller G(s) are $\{-2, -3, A_k\}$.

For the same example, the following table shows what can be achieved by static K(s), that is by an appropriate choice of D_k.

pole to be removed	D_k	n_g	closed-loop poles
−2	−5	1	{−1, −3, −2}
−1	−4	0	{−2, −2}
−3	−4	0	{−2, −2}

We now address the case when the parameter K(s), between the measurement residual (z_p-z_j) and the control u_k is dynamic. Appending additional dynamics K(s) to the original controller J(s), increases the dimension of the overall controller Q(s) to $n_q=n_p+n_k$, where n_q is the dimension of Q(s) and n_k is that of K(s). Recalling the properties of observer-based controllers with respect to the closed loop poles, the addition of the parameter K(s) does not change the eigenvalues of the closed loop system but just adds to them. This implies that the internal stability of the closed loop system depends on the stability of K(s). Thus by choosing a proper stable K(s) one can guarantee the stability of the closed loop system. The necessity of satisfying the orthogonality condition between the left regulator and right observer eigenvectors corresponding to the eigenvalues to be made uncontrollable and unobservable respectively, makes it difficult to determine a static feed through loop that would yield a reduced order controller. However, controller reduction by observer based controller parametrization may be achieved when the parameter matrix, K(s), is dynamic. This section extends the results obtained for the existence of an order reducing static feed-through loop to the dynamic case.

Theorem 9.4 states that the necessary and sufficient condition for the existence of the parameters of K(s) to achieve non-minimality of the overall controller Q(s) is the simultaneous satisfaction of equations (9-42a) and (9-42b). The following theorem states a necessary and sufficient condition for the existence of the parameters of K(s) that will simultaneously satisfy equations (9-42a) and (9-42b).

Theorem 9.6: *Under the assumption that $\Lambda_{uc} \cap \Lambda_{uo} = \phi$, the parameters of K(s) simultaneously satisfying equations (9-42a) and (9-42b) exist if and only if*

$$\Phi_r\Psi_o + \Phi_k\Psi_k = 0 \qquad (9\text{-}49)$$

where Φ_k and Ψ_k are arbitrary.

Proof : Assume $\Lambda_{uc} \cap \Lambda_{uo} = \phi$. Then premultiplying Eqn.(9-42a) by $[\Phi_k \; \Phi_r B_p]$, yields

$$\Phi_k \Psi_k \Lambda_{uo} - \Phi_r B_p C_j \Psi_o = \Phi_k \Psi_k \Lambda_{uc} + \Phi_r B_j C_p \Psi_o \tag{9-50a}$$

Adding and subtracting A_p, and using Eqns.(9-43a)and (9-43b) results in

$$[\Phi_r \Psi_o + \Phi_k \Psi_k]\Lambda_{uo} - \Lambda_{uc}[\Phi_r \Psi_o + \Phi_k \Psi_k] = 0 \tag{9-50b}$$

The solution to equation (9-50b) under the hypothesis of the theorem yields (9-49).

Recall, that the existence of an order reducing static feed-through loop is guaranteed under the satisfaction of an orthogonality condition between the left regulator and right observer eigenvectors associated with the removable eigenvalues. This necessary and sufficient condition is restrictive in the sense that, rarely can one find systems which would satisfy this orthogonality condition. However, the necessary and sufficient condition stated according to Theorem 9.6. can almost always be satisfied indicating that one could almost always determine a dynamic feed through loop that would yield a reduced order controller. This is attributed to the arbitrariness of Φ_k and Ψ_k.

Theorem 9.6 indicates that it is possible to simultaneously render some observer eigenvalues unobservable and some regulator eigenvalues uncontrollable. Having established the existence of dynamic feed through loops that would yield reduced order controllers, the next logical step would be the computation of the parameter matrix K(s). Theorem 9.7 presents a closed form solution to the equations (9-42a) and (9-42b). This result is an extension of the case when K(s) is static, and can be proved similarly.

Theorem 9.7: *The solution $\{A_k, B_k, C_k, D_k\}$ to the necessary and sufficient conditions equations is*

$$\underline{\mathbf{A}}_k = \underline{\mathbf{C}}_j \underline{\mathbf{C}}_p^- + \underline{\mathbf{B}}_p^- \underline{\mathbf{B}}_j - \underline{\mathbf{B}}_p^- \underline{\mathbf{B}}_p \underline{\mathbf{C}}_j \underline{\mathbf{C}}_p^- + [I - \underline{\mathbf{B}}_p^- \underline{\mathbf{B}}_p] Z [I - \underline{\mathbf{C}}_p \underline{\mathbf{C}}_p^-] \tag{9-51}$$

where

$$\underline{\mathbf{A}}_k = \begin{bmatrix} A_k & B_k \\ C_k & D_k \end{bmatrix} \quad \underline{\mathbf{C}}_j = \begin{bmatrix} \Psi_k \Lambda_{uo} \\ -C_j \Psi_o \end{bmatrix} \quad \underline{\mathbf{C}}_p = \begin{bmatrix} \Psi_k \\ -C_p \Psi_o \end{bmatrix}$$

$$\underline{\mathbf{B}}_p = \begin{bmatrix} \Phi_k & \Phi_r B_p \end{bmatrix} \quad \underline{\mathbf{B}}_j = \begin{bmatrix} \Lambda_{uc} \Phi_k & -\Phi_r B_j \end{bmatrix} \tag{9-52}$$

where $(\cdot)^-$ denotes generalized inverses, and Z is arbitrary.

The family of dynamic K(s) constructed according to Theorem 9.7, will yield a reduced order controller of dimension

$$n_g = n_p - (n_{uo} + n_{uc}) + n_k \tag{9-53}$$

where n_k is the order of the parameter matrix K(s).

From the properties of observer based controllers, the resulting closed loop system will always retain the characteristic values of the parameter matrix K(s). i.e ., the resulting closed loop system will be stable if and only if A_k is stable. To pursue this issue consider

$$A^* = \underline{C}_j \underline{C}_p^{-} + \underline{B}_p^{-} \underline{B}_j - \underline{B}_p^{-}\underline{B}_p\underline{C}_j\underline{C}_p^{-} \tag{9-54}$$

$$B^* = [I - \underline{B}_p^{-}\underline{B}_p] \quad ; \quad C^* = [I - \underline{C}_p\underline{C}_p^{-}] \tag{9-55}$$

Then Eqn. (9-51) can be rewritten as

$$\underline{A}_k = A^* + B^* \, Z \, C^* \tag{9-56}$$

Notice that equation (9-56) has a structure similar to that of an output feedback problem, where Z is a free parameter to be chosen such the eigenvalues of A_k are stable. However, since A_k is only a submatrix - namely, the (1,1) submatrix - of A^*, the corresponding submatrix in Eqn.(9-56) need be considered as an output feedback problem. Thus a the free parameter Z which would guarantee a stable A_k will exist if and only if the corresponding submatrix pair of (A^*,B^*) is stabilizable, and the corresponding submatrix pair of (C^*,A^*) is detectable. More stringent conditions under which one could guarantee the stability of K(s) are yet to be determined.

Besides stability, performance of the resulting closed loop system is an equally important issue in controller reduction problems. Both Theorem 9.6, and 9.7 seem to indicate that the choice of removable eigenvalues play an important role in the computation of the parameters of K(s). One could employ one of several existing schemes, such as balancing [31] and component cost analysis [45], to determine the "removable" eigenvalues. Another way of approaching the optimization problem is as follows: one could determine an optimal controller K*(s), using standard parametric optimization approach. We could then select a K(s) which would best approximate the optimal controller by (say) solving a suitable minimum norm problem. The arbitrariness of Z in equation (9-51) allows us some flexibility in obtaining a stable K(s) that would result in a reduced order controller which would best approximate the full order controller in terms of a specified performance.

9.3 Conclusions

It has been shown that by introducing a static feed-through loop between the measurement residual and the control signal, the order of a given observer-based controller can be reduced while guaranteeing the stability of the closed loop system. This is achieved by making some eigenvalues of the controller uncontrollable and/or unobservable. The necessary and sufficient conditions for the existence of such feed-through loops have been characterized, thereby parametrizing such loops. These results have been extended to the case when K(s) is dynamic, however, stability of the resulting closed loop system is still an unresolved issue. With the parametrization accomplished, problems can now be posed to address the performance issues.

CHAPTER 10

DOMINANT MODES OF MECHANICAL SYSTEMS

Recall from Chapter 9 that in order to incorporate the parametrization of order-reducing K(s) in a design procedure requires the identification of poles that are to be rendered uncontrollable and unobservable. This chapter deals with identifying modes of a mechanical system that are dominant in an L_2 sense. Specifically, it considers obtaining a reduced model of a stable mechanical system with proportional damping. Such systems can be conveniently represented in modal coordinates. The popular scheme, Modal Cost Analysis (MCA) of [61], offers a simple means of identifying dominant modes for retention in the reduced model. In MCA, the dominance is measured via the modal costs. Though this measure leads to simple computations, it does not exactly reflect the more appropriate model error which is the L_2 norm of the output-error between the full and the reduced models. Normally, the model error is computed *after* the reduced model is obtained, since it is believed that, in general, the model error cannot be easily computed a priori. The main thrust of this note is to point out that the model error can also be calculated a priori, just as easily as the modal costs. Hence the model error itself can be used to determine the dominant modes. Moreover, the simplicity of the computations do not presume any special properties of the system, such as small damping, orthogonal-symmetry, etc. The development presented herein can be seen as a specialization of that in [62], to mechanical systems.

10.1 Problem Formulation

Consider a time-invariant mechanical system, described in its physical coordinates q, given below.

$$\begin{aligned} M\,\ddot{q}(t) + D\,\dot{q}(t) + K\,q(t) &= B\,u(t) \\ y(t) &= Cq(t) + \underset{\sim}{C}\dot{q}(t) \end{aligned} \tag{10-1}$$

where $q \in \mathbf{R}^n$, $u \in \mathbf{R}^m$, $y \in \mathbf{R}^k$; u is assumed to be a Gaussian white noise process with unit intensity. Both the mass matrix M and the stiffness matrix K are assumed to be symmetric and positive definite. The dissipation matrix D is assumed to arise from

proportional damping (not necessarily small) such that the above system is asymptotically stable. Under these assumptions, the model (10-1) can be equivalently expressed in its modal coordinates as

$$\ddot{\eta}_i + 2\zeta_i\omega_i\dot{\eta}_i + \omega_i^2\eta_i = b_i^T u \;\; ; \;\; i = 1,2, \ldots, n \tag{10-2a}$$

$$y = \sum_{i=1}^{n} (c_i\eta_i + \underline{c}_i\dot{\eta}_i) \tag{10-2b}$$

where ω_i and ζ_i are the usual i^{th} natural frequency and the corresponding damping ratio, respectively.

When n is large, one is faced with the problem of model reduction to facilitate a subsequent analysis and control design. Quite often, a reduced model of the system (10-2a&b) is obtained by retaining some r modes of the total n modes – a process called herein as a modal reduction. In order to produce an acceptable reduced model, the question that needs to be answered is "which r of the n modes should be retained ?" Using the notation N_n for the integer set $\{1,2,...,n\}$, the issue in modal reduction is to identify an r-element "reduction-set" $N_{red} \subset N_n$, such that the following model

$$\ddot{\eta}_i + 2\zeta_i\omega_i\dot{\eta}_i + \omega_i^2\eta_i = b_i^T u \;\; ; \;\; i \in N_{red} \tag{10-3a}$$

$$y_r = \sum_{i \in N_{red}} (c_i\eta_i + \underline{c}_i\dot{\eta}_i) \tag{10-3b}$$

would be an acceptable reduced model. Defining the error e(t) between the full model (10-2a&b) and the reduced model (10-3a&b) as $e(t) = y(t) - y_r(t)$, an acceptable reduced model would minimize the model error δV defined below.

$$\delta V = \lim_{t\to\infty} E\left\{\frac{1}{t}\int_0^t e^T(\sigma)e(\sigma)\,d\sigma\right\} \tag{10-4}$$

10.2 Dominant Modes by Modal Cost Analysis

For an arbitrary system, not necessarily a mechanical system, the model error δV associated with its reduced model is normally computed after the reduced model is obtained. Consequently, obtaining an acceptable reduced model for such systems becomes an iterative process. In order to simplify the reduction process, methods have been

proposed to employ different criteria instead of (10-4). One such method is the Component Cost Analysis (CCA) of Skelton [45]. The CCA attempts to use the cost error ΔV defined below, as its criterion for model reduction.

$$\Delta V = V - V_r , \tag{10-5a}$$

where

$$V = \lim_{t \to \infty} E\left\{\frac{1}{t}\int_0^t y^T(\sigma) y(\sigma)\, d\sigma\right\}, \tag{10-5b}$$

$$V_r = \lim_{t \to \infty} E\left\{\frac{1}{t}\int_0^t y_r^T(\sigma) y_r(\sigma)\, d\sigma\right\}. \tag{10-5c}$$

Now, the reduced models which are optimal in the sense of minimizing the model error δV, are known to satisfy the orthogonal property [46]

$$\langle e, y_r \rangle = \lim_{t \to \infty} E\left\{\frac{1}{t}\int_0^t e^T(\sigma) y_r(\sigma)\, d\sigma\right\} = 0. \tag{10-6}$$

Since the model error δV can be expressed in terms of the cost error ΔV and the inner product $\langle e, y_r \rangle$ as given below

$$\delta V = \Delta V - 2\langle e, y_r \rangle , \tag{10-7}$$

it follows that the cost error is an appropriate criteria to use, provided the resulting reduced model is known to be near optimal, if not optimal. Though the computation of ΔV is simpler than that of δV, it still requires the availability of the reduced model. In view of this, the CCA employs the predicted cost error $\widehat{\Delta V}$, defined below, for its model reduction decisions.

$$\widehat{\Delta V} = V - \widehat{V}_r , \tag{10-8}$$

where $\widehat{V}_r$ is a prediction of V_r. With x_i defined as the (first order) state representing the i^{th} component, $\widehat{V}_r$ is computed according to

$$\widehat{V}_r = \sum_{i \in N_{red}} \widehat{V}_i \tag{10-9a}$$

and where $\hat{V}_i$, the i^{th} component cost, is defined as [45]

$$\hat{V}_i = \frac{1}{2} \lim_{t\to\infty} E\left\{\frac{1}{t}\int_0^t \frac{\partial[y^T(\sigma)y(\sigma)]}{\partial x_i(\sigma)} x_i(\sigma)\, d\sigma\right\} \tag{10-9b}$$

Since the calculation of the component costs requires only the full model, the predicted cost error $\widehat{\Delta V}$ can be computed before a reduced model is obtained.

The application of the CCA to models represented in modal coordinates, such as in Eqs. (10-2a and b), is called the Modal Cost Analysis (MCA), see [61]. Therefore, modal reduction by MCA chooses an N_{red} such that the predicted cost error $\widehat{\Delta V}$ is minimized. Except under special cases, there is no guarantee that the predicted cost error $\widehat{\Delta V}$ equals the model error δV. It turns out that, for any choice of N_{red}, the corresponding δV can be computed just as easily as computing $\widehat{\Delta V}$. Hence, one may use the more appropriate criterion δV for choosing N_{red}. These details are developed below.

10.3 Closed Form Expression for Model Error

Provided all the modes are observable and controllable, truncation of any mode would affect the output y. To determine such affect, define

$$\psi_i = c_i\eta_i + \underline{c}_i\dot{\eta}_i \quad \text{so that} \quad y = \sum_{i\in N_n} \psi_i \tag{10-10}$$

and call ψ_i as the "i^{th} modal-output." By defining the "truncation-set" $N_{tru} = N_n - N_{red}$, the following expressions result:

$$e = \sum_{i\in N_{tru}} \psi_i \tag{10-11a}$$

$$\delta V = \sum_{i\in N_{tru}} \left[\sum_{j\in N_{tru}} \Psi_{ij}\right] \tag{10-11b}$$

where

$$\Psi_{ij} = \lim_{t\to\infty} E\left\{\frac{1}{t}\int_0^t \psi_i^T(\sigma)\psi_j(\sigma)\, d\sigma\right\};\; i,j \in N_n \tag{10-11c}$$

In the spirit of CCA, call Ψ_{ij} the cost-correlation between the modes i and j. Note that Ψ_{ij}

measures the correlation between the i^{th} and the j^{th} modal-outputs, and depends only on the i^{th} and the j^{th} modal data. The expression for computing Ψ_{ij} for all $i,j \in N_n$, is given in the next section. Given the modal data, these cost-correlations can be computed for all the modes, and the model error δV can be determined for any r-element reduction-set $N_{red} \subset N_n$ from the expression (10-11b). Notice that the model error can be computed *a priori* by simple summation of appropriate Ψ_{ij}s.

Clearly, there are $[n! / \{r!(n-r)!\}]$ number of possible reduction-sets, of which one will yield the smallest model error. To facilitate the selection of the best reduction-set, construct an (n,n) "cost-correlation-matrix" Ψ whose (i,j) element equals Ψ_{ij}, and define an operation $\Sigma\Sigma : \mathbf{R}^{(n,m)} \to \mathbf{R}$ for arbitrary n and m by $\Sigma\Sigma(Z) = \sum_{i=1}^{n} \sum_{j=1}^{m} Z_{ij}$. Then, the cost-correlation-matrix possesses these properties for any N_{red}:

1) $\delta V = \Sigma\Sigma(\Psi_{tru})$ (10-12a)

2) $V = \Sigma\Sigma(\Psi)$ (10-12b)

3) $V_r = \Sigma\Sigma(\Psi_{red})$ (10-12c)

4) $\langle e, y_r \rangle = \Sigma\Sigma(\Psi_{retr})$ (10-12d)

5) $\hat{V}_i = \Sigma\Sigma(\Psi_i)$ (10-12e)

where

$$\Psi_{tru} = [\, \Psi_{ij};\ i,j \in N_{tru} \,],\ \text{an } (n-r,n-r) \text{ submatrix of } \Psi, \qquad (10\text{-}13a)$$

$$\Psi_{red} = [\, \Psi_{ij};\ i,j \in N_{red} \,],\ \text{an } (r,r) \text{ submatrix of } \Psi, \qquad (10\text{-}13b)$$

$$\Psi_{retr} = [\, \Psi_{ij};\ i \in N_{red},\ j \in N_{tru}],\ \text{an } (r,n-r) \text{ submatrix of } \Psi, \qquad (10\text{-}13c)$$

$$\Psi_i = [\, \Psi_{ij};\ j \in N_n],\ \text{an } (1,n) \text{ matrix (i.e., } i^{th} \text{ row) for all } i. \qquad (10\text{-}13d)$$

Notice that the fourth property of the cost-correlation-matrix allows one to determine a suboptimality index [63] associated with the reduced model. This index can be used to determine if the reduced model could be improved by further optimizations. It measures how closely the orthogonality condition (10-6) is satisfied. The second and third properties above provide means of computing the quadratic cost functions V and V_r associated with the full and the reduced models, respectively. Moreover, the three errors, namely, the

model error δV, the cost error ΔV, and the predicted cost error $\widehat{\Delta V}$, are related through the property (10-12d) as follows

$$\Delta V = \delta V + 2\,\Sigma\Sigma(\Psi_{retr}), \tag{10-14a}$$

$$\widehat{\Delta V} = \delta V + \Sigma\Sigma(\Psi_{retr}). \tag{10-14b}$$

Hence, unless $\Sigma\Sigma(\Psi_{retr}) = 0$, modes determined as dominant based upon either the cost error or the predicted cost error may not yield the smallest model error. Since the expression (10-11b) is available for the computation of the model error for any choice of N_{red}, dominance of modes can be established based on δV itself. The disadvantage, however, is in having to compute $[(n-r)^2-1]$ additions $[n!\,/\,\{r!(n-r)!\}]$ times.

Theorem 10.1: *The expression for computing Ψ_{ij} for all $i,j \in N_n$, is given below.*

$$\Psi_{ij} = X_{ij}^{11}[\, c_i^T c_j - \frac{\Delta_{ij}}{\delta_{ij}}(\underline{c}_i^T c_j - c_i^T \underline{c}_j) - \frac{\Delta_{ij}^2}{\delta_{ij}^2}\underline{c}_i^T \underline{c}_j \,] + \frac{b_i^T b_j}{\delta_{ij}}\underline{c}_i^T \underline{c}_j \tag{10-15a}$$

where

$$X_{ij}^{11} = \frac{(b_i^T b_j)\,\delta_{ij}}{\Delta_{ij}^2 + \omega_i^2 \delta_{ij}^2 - 2\zeta_i \omega_i \delta_{ij} \Delta_{ij}}, \tag{10-15b}$$

$$\Delta_{ij} = \omega_i^2 - \omega_j^2, \text{ and } \delta_{ij} = 2\zeta_i\omega_i + 2\zeta_j\omega_j. \tag{10-15c}$$

Proof: Defining the i^{th} modal states as $x_i^T = [\,\eta_i\ \dot{\eta}_i\,]$, it follows from the definition of the cost-correlation Ψ_{ij} and the modal-outputs ψ_i that

$$\Psi_{ij} = \lim_{t\to\infty} E\left\{\frac{1}{t}\int_0^t x_i^T(\sigma) C_i^T C_j x_j(\sigma)\, d\sigma\right\} = \mathrm{trace}\{C_i^T C_j X_{ji}\}, \tag{10-16a}$$

where

$$X_{ij} = \lim_{t\to\infty} E\left\{\frac{1}{t}\int_0^t x_i(\sigma) x_j^T(\sigma)\, d\sigma\right\}, \tag{10-16b}$$

$$C_i = [\, c_i \ \underline{c}_i \,] \,. \tag{10-16c}$$

The steady state correlation matrix between the i^{th} and the j^{th} modal states is given by

$$X_{ij} = \begin{bmatrix} X_{ij}^{11} & X_{ij}^{12} \\ X_{ij}^{21} & X_{ij}^{22} \end{bmatrix} \tag{10-17a}$$

where

$$\begin{cases} X_{ij}^{12} = -X_{ij}^{21} = \dfrac{\Delta_{ij}}{\delta_{ij}} X_{ij}^{11} \\ X_{ij}^{22} = \dfrac{b_i^T b_j}{\delta_{ij}} - \dfrac{\Delta_{ij}}{\delta_{ij}} X_{ij}^{12} \end{cases} , \tag{10-17b}$$

with X_{ij}^{11} as defined above in Eqn.(10-15b). Substitution of the Eqs. (10-17a) and (10-17b) into Eqn.(10-16a) completes the proof.

Expressions similar to (10-17a &b) can be found in several references, see [64] for example. The expression (10-15a) is valid for *all* i *and* j, regardless of whether the frequencies are repeated or whether the damping is small. Of course, the expression simplifies under special cases, as shown below.

a) If $\omega_i = \omega_j$, then $\Psi_{ij} = \dfrac{b_i^T b_j}{2\omega_i^3(\zeta_i+\zeta_j)} \{ c_i^T c_j + \omega_i^2 \underline{c}_i^T \underline{c}_j \}$ (10-18a)

b) If $\omega_i \neq \omega_j$, and both $\zeta_i, \zeta_j \rightarrow 0$, then $\Psi_{ij} \rightarrow \dfrac{b_i^T b_j}{\Delta_{ij}} \{ \underline{c}_i^T c_j - c_i^T \underline{c}_j \}$ (10-18b)

c) If $\omega_i = \omega_j$, and both $\zeta_i, \zeta_j \rightarrow 0$, then $\Psi_{ij} \rightarrow \dfrac{b_i^T b_j}{4\zeta_i \omega_i^3} \{ c_i^T c_j + \omega_i^2 \underline{c}_i^T \underline{c}_j \}$ (10-18c)

Note by the special case (b) above that for a lightly damped system (such as a flexible space structure) with distinct frequencies and with either (i) $b_i^T b_j = 0$ for all i and j, or (ii) $\underline{c}_i^T c_j = 0$ for all i and j, one obtains $\Psi_{ij} \rightarrow 0$ for all $j \neq i$, and for all i. It therefore follows from Eqns.(10-8,10-9a,10-11b,10-12b,10-12e) that $\widehat{V}_i = \Psi_{ii}$, $\delta V = \sum_{i \in N_{tr}} \Psi_{ii}$, and $\delta V = \widehat{\Delta V}$. Hence, for systems satisfying these assumptions, reduced models

obtained by MCA, which are guaranteed to minimize the predicted cost error $\widehat{\Delta V}$, also minimize the model error δV.

10.4 Examples

Two examples are provided now to show that reduced models generated as suggested in this note will be at least as good as, if not better than, those produced by Modal Cost Analysis. In both the examples, we begin with five modes and produce reduced models of different orders. The reduced models are evaluated in terms of their model errors, and the results are tabulated below. The two examples differ only in the frequency of the second mode, and in the damping ratios (the damping ratios are assumed to be equal for all the modes): in Example 1, $\omega_2 = 2$, and $\zeta = 0.075$; in Example 2, $\omega_2 = 1$, and $\zeta = 0.005$. The parameters common to both the examples are: $\omega_1 = 1$, $\omega_3 = 9$, $\omega_4 = 16$, $\omega_5 = 25$, $b_1 = 0.9877$, $b_2 = -0.309$, $b_3 = -0.891$, $b_4 = 0.5878$, $b_5 = 0.707$, $c_1 = 0.9877$, $c_2 = 0.309$, $c_3 = -0.891$, $c_4 = -0.5878$, $c_5 = 0.707$, and $\underline{c}_i = 0$ for all i.

Table 10.1 Model error comparison for Example 1 ($\omega_2 = 2$, $\zeta = 0.075$)

	based on modal costs		based on model error	
r	retained modes	model error	retained modes	model error
1	{1}	6.7271e-03	{1}	6.7271e-03
2	{1,3}	3.9368e-03	{1,2}	2.9686e-03
3	{1,3,5}	3.8985e-03	{1,2,3}	1.3641e-04
4	{1,3,4,5}	3.7986e-03	{1,2,3,4}	5.3301e-05

Table 10.2 Model error comparison for Example 2 ($\omega_2 = 1$, $\zeta = 0.005$)

	based on modal costs		based on model error	
r	retained modes	model error	retained modes	model error
1	{1}	5.0131e-01	{1}	5.0131e-01
2	{1,3}	4.5809e-01	{1,2}	4.5479e-02
3	{1,3,4}	4.5663e-01	{1,2,3}	2.2557e-03
4	{1,3,4,5}	4.5583e-01	{1,2,3,4}	7.9952e-04

10.5 Conclusions

The quality of a reduced model is often computed through the model error. In general this model error can be computed only after the reduced model is obtained. This chapter points out that for mechanical systems represented in modal coordinates, the model error can be computed a priori for any choice of retained modes. In fact, the computations involved are just as simple as those needed for the general Modal Cost Analysis. When the system is known to satisfy certain assumptions, it is shown that the reduced models produced by Modal Cost Analysis, which minimize the predicted cost error, actually minimize the model error. However, such is not the case in general. In order to facilitate the selection of the set of retained modes that would actually minimize the model error, a cost-correlation-matrix has been presented. The model error is computed by simple summation of the appropriate elements of this matrix. Though the cost-correlation-matrix is constructed from the given modal data only once, the computational effort may still become formidable depending upon the number of the original modes and the number of the retained modes.

11. CONCLUSIONS AND WORK FOR FUTURE RESEARCH

11.1 Conclusions

We revealed several important eigen properties of the stabilizing solutions for both DGKF and GD paremeter dependent H^∞ Riccati equations. Among them, the most prominent ones are: (1) $\rho[X_\infty(\gamma)Y_\infty(\gamma)]$ is a nonincreasing, convex function of γ on $(\beta,+\infty)$. (2) $\lambda_{min}[\tilde{X}(\gamma)]$ is a nondecreasing, concave function of γ on $(\alpha_x,+\infty)$. (3) $\tilde{X}(\gamma)$ is invertible almost everywhere on $(\alpha_x,+\infty)$. Based on these properties, quadratically convergent algorithms were developed to compute the γ_∞, the optimal H^∞ norm, such that $\rho[X_\infty(\gamma_\infty)Y_\infty(\gamma_\infty)] = \gamma_\infty^2$ and to compute β, the infimum of γ such that the two H^∞ Riccati equations have positive semidefinite stabilizing solutions. These algorithms are much faster than the other existing search schemes.

In the proof of the convexity and monotonicity properties, especially for the general case with D_{11} nonzero (i.e., the GD case), the H^∞ Riccati Hamiltonains are very complicated functions of γ and hence make the investigation of the properties extremely difficult. In this report, several new decomposition techniques and some key intermediate variables are employed to tackle the complexities.

The formulation of the standard H^∞ optimization problem, an easy way of constructing a state-space realization of the generalized plant, and a modified version of Glover and Doyle formulas for constructing an optimal controller were addressed in the report. No numerical difficulty will arise in constructing an H^∞ optimal controller if we are allowed a proper controller. In most applications, we may like to have a strictly proper controller with limited bandwidth. In this case, a trade-off between the H^∞ performance and the bandwidth should be made by degrading the H^∞ norm from its optimum in order to reduce the controller bandwidth.

Using higher order D(s) for curve fitting in the D-step of D-K iteration will give better robust performance but higher order controller. In the fighter aircraft longitudinal flight control μ-synthesis design example, we found that the higher order controller can be easily reduced to the order of the plant by well known model reduction techniques like balanced realization truncation or Hankel approximation without degrading much of the closed-loop robust performance.

Controller reduction is important especially when a gain scheduling is required in

the implementation of controllers. In this report, we reduced the order of the controller with maintaining the closed-loop input-output relationship in mind. Three approaches are proposed to achieve this objective. The first is a structured truncation method in which a structured transformation on the closed-loop system was used to minimize a cost function, which is defined in terms of the controllability and observability gramians of the closed-loop system. The examples showed that the structured truncation based on the closed-loop controllability and observability gramians is quite appealing. For the general controller other than the H_2 optimal controller, this concept still works although the computation is more complicated.

The second controller reduction approach proposed is developed based on the observer-based controller parametrization. We showed that the poles of the closed-loop system with the observer-based controller parametrization are the regulator poles, the observer poles, together with the poles of the added stable parameter matrix. If the controller is realized by a minimal realization, the closed loop poles will include all the poles of the added stable parameter matrix and a subset of the regulator and the observer poles. The parameter matrix is chosen such that the controller realization is non-minimal, thereby permitting a minimal realization of the controller to serve as a reduced order controller.

The basic concept of the third controller reduction approach is similar to the second one. The difference is that the third approach employs the H^∞ controller parametrization and attempts to keep the H^∞ norm of the reduced closed-loop system below a prescribed value in addition to retaining the closed-loop stability. In this report we presented a direct design algorithm for $(n-p_2)$th order H^∞ controller, where n is the order of the generalized controller and p_2 is the number of independent measured outputs. A set of equations were derived by which a parameter matrix can be constructed such that the realization of the controller obtained from DGKF formulas [7] is not minimal and the H^∞ norm of the closed-loop system is less than a prescribed value. By deleting the unobservable and/or uncontrollable states of the controller, $(n-p_2)$th order H^∞ controller can be obtained.

11.2 Work for Further Research

Most of the plant uncertainties are structured. The uncertain system can be reconfigurated as an M-Δ structure in which the uncertainties Δ has a block diagonal structure and is separated from the rest of the system. A robust control problem can be formulated as a minimization problem called μ synthesis problem (or structured singular

value design problem) based on the M-Δ structure. Although the μ synthesis problem is a practical formulation, the problem is very difficult to solve. The minimization problem is not a convex problem and therefore no algorithm is available for searching for the global minimum. Yet, an algorithm called D-K iteration has been employed in μ-synthesis to find a local minimum. In the future research, we will investigate the formulation of the μ-synthesis problem and the computational issues arising in solving the problem.

The order of the controller designed via μ-synthesis can be arbitrarily high. Based on our preliminary observation, we found that the controller's order can be greatly reduced to that of the generalized plant almost without deteriorating the closed-loop performance. The theory behind this fact can be interesting. Controller reduction problem for μ-synthesis controller is an important research problem.

The structured truncation method and H^∞ controller parametrization are two potential controller reduction tools for μ-synthesis controllers. The structured truncation method has been showed very effective in reducing the order of H^2 controllers. It can be also applied to H^∞ and μ-synthesis controllers, although the computations involved are much more complicated. Some computational issues arising in this controller reduction approach still remain unsolved.

In this research period, we have successfully found an easy way to construct a set of an (n-p2)th order stabilizing controllers by H^∞ controller parametrization. We also established a group of equations by which an (n-p2)th order H^∞ stabilizing controller can be obtained. However, this group of equations in general are difficult to solve. More effort is required in the line of this research.

Aircraft flight dynamics is nonlinear and flight-condition dependent. Our previous research assumed that a local linearized model for each selected point inside the flight envelope is available, and concentrated on designing a local linear controller for each linearized model. In the future research, we will consider the full envelope design and investigate the controller gain scheduling, i.e., the parametrization of these controllers in terms of flight conditions.

12. REFERENCES

[1] B. A. Francis, *A Course in H^∞ Control Theory*, Springer-Verlag, 1987.

[2] J. Doyle, *Lectures Notes, ONR/Honeywell Workshop on Advances in Multivariable Control*, Minneapolis, Minnesota, Oct. 1984.

[3] H. Kwakernaak, "Minimax Frequency Domain Performance and Robustness Optimization of Linear Feedback Systems," *IEEE Transactions on automatic control*, Vol. AC-30, Oct. 1985, pp.994-1004.

[4] J. Doyle, "Analysis of Feedback Systems with Structured Uncertainties," *IEE Proceedings*, Vol. 129 Pt.D, No.6, 1982, pp. 242-250.

[5] A. K. Packard, "What's New with μ: Structured Uncertainty in Multivariable Control", *Ph.D. Dissertation*, U.C. Berkeley, 1988.

[6] J. C. Doyle, A. K. Packard, and K. Zhou, "Review of LFT's, LMI's, and μ," *Proceedings of 30th Conference on Decision and Control*, Dec. 1991.

[7] J. C. Doyle, K. Glover, P.P. Khargonekar, and B.A. Francis, "State-space Solutions to Standard H_2 and H_∞ Control Problems," *IEEE Transactions on Automatic Control*, Vol. AC-34, No. 8, Aug. 1989, pp.831-847.

[8] K. Glover and J. Doyle, "State-space formulae for all stabilizing controllers that satisfy an L_∞-norm bound and relations to risk sensitivity," *Systems Control Letters*, Vol. 11, pp. 167-172, 1988.

[9] G. J. Balas, J. C. Doyle, K. Glover, A. Packard, and R. Smith, *μ-Analysis and Synthesis Toolbox*, MUSYN Inc. and The Math Works, Inc., 1991.

[10] R. Y. Chiang and M. G. Safonov, *Robust-Control Toolbox*, The Math Works, Inc., 1988.

[11] J. C. Doyle and G. Stein, "Multivariable Feedback Design: Concepts for a Classical/Modern Synthesis," *IEEE Transactions on automatic control*, Vol. AC-26, Feb. 1981, pp.4-16.

[12] M.G. Safonov, D.J.N. Limebeer, and R.Y. Chiang, "Simplifying the H^∞ Theory via

Loop Shifting, Matrix-pencil and Descriptor Concepts," Int. J. Control, vol. 50, no. 6, pp2467-2488, 1989.

[13] P. Pandey, C. Kenny, A. Laub, and A. Packard, "Algorithms for computing the optimal H_∞-norm, "*Proceedings of the 29th Conference on Decision and Control*, Dec. 1990.

[14] B. C. Chang, X. P. Li, S. S. Banda, and H. H. Yeh, "Iterative computation of the optimal H_∞ norm by two-Riccati-equtions method," *Proceedings of the 29th Conference on Decision and Control*, Dec. 1990.

[15] C. Scherer, "H_∞-control by state-feedback and fast algorithms for the computation of optimal H_∞-norms," *IEEE Transactions on Automatic Control*, Vol. AC-35, pp. 1090-1099, 1990.

[16] X. P. Li, and B. C. Chang," On convexity of H^∞ Riccati solutions and its applications," *IEEE Transactions on Automatic Control*, Vol. AC-38, pp. 963-966, June 1993.

[17] H. Kwakernaak and R. Sivan, *Linear Optimal Control Systems*, John Wiley & Sons, Inc., 1972.

[18] I. Postlethwaite, D.-W. Gu, and S. D. O'Young, "Some computational results on size reduction in H_∞ design", *IEEE Trans. Automat. Contr.*, Vol. 33, pp. 177-185, 1988.

[19] T. Kailath, *Linear Systems*, Englewood Cliffs, NJ: Prentice-Hall, 1980.

[20] W. M. Wonham, *Linear Multivariable Control: A Geometric Approach*, 3rd ed., N.Y., Springer-Verlag, 1985.

[21] T. Kato, *Perturbation Theory for Linear Operators*, New York: Springer-Verlag, 1984.

[22] X. P. Li, B. C. Chang, "On Convexity of H^∞ Riccati Solutions," *Proceedings of the 30th Conference on Decision and Control*, Dec. 1991.

[23] A. Wouk, "A Course of Applied Functional Analysis", John Wiley and Sons, Inc., New York, 1979.

[24] B.C. Chang, X.P. Li, H.H. Yeh, and S.S. Banda, "Computation of the H_∞ Norm of a Transfer Function", *Proceedings of the 1990 American Control Conference*, May 1990.

[25] S. Boyd and V. Balakrishnan, "A Regularity Result for the Singular Values of a Transfer Matrix Quadratically Convergent Algorithm for Computing its L_∞ Norm", *Proceedings of the 28th Conference on Decision and Control*, Dec. 1989.

[26] R. S. Sezginer and M. L. Overton, " The Largest Singular Value of $e^x A_0 e^{-x}$ is Convex on Convex Sets of Commuting Matrices," *IEEE Transactions on Automatic Control*, AC-35, pp.229-230, Feb. 1990.

[27] X. P. Li, B. C. Chang, S. S. Banda, H. H. Yeh, "Robust Control System Design Using H^∞ Optimization Theory," *Journal of Guidance, Control, and Dynamics*, scheduled to be published in Sept., 1992.

[28] K. Glover, "All Optimal Hankel-norm Approximations of Linear Multivariable Systems and Their L^∞-error Bounds," *International Journal of Control*, Vol. 39, No. 6, 1984, pp.1115-1193.

[29] B. D. O. Anderson and Y Liu "Controller reduction: concepts and approaches," *IEEE Trans. Automat. Contr.*, vol. AC-34, pp. 802-812, Aug. 1989.

[30] D. S. Bernstein and D. C. Hyland, "The optimal projection equations for fixed-order dynamic compensation," *IEEE Trans. Automat. Contr.*, vol. AC-29, pp.1034-1037, Nov. 1984.

[31] B. C. Moore, "Principal component analysis in linear systems: Controllability, observability, and model reduction," *IEEE Trans. Automat. Contr.*, vol. AC-26, no. 1. pp. 17-32, 1981.

[32] L. Pernebo and L. M. Silverman, "Model reduction via balanced state space representations," *IEEE Trans. Automat. Contr.*, vol. AC-27, no. 2, pp. 382-387, 1982.

[33] A. Yousuff and R. E. Skelton, "A note on balanced controller reduction." *IEEE Trans. Automat. Contr.*, vol. AC-29, no. 3, pp. 254-257, 1984.

[34] V. M. Adamjan, D. Z. Arov, and M. G. Krein, "Analytic properties of Schmidt pairs

for Hankel operator and the generalized Schur-Takagi problem," *Math. USSR Sbornik*, vol. 15, no. 1, pp. 31-73, 1971.

[35] A. Yousuff and R. E. Skelton, "Covariance equivalent realizations with application to model reduction of large-scale systems," *Contr. Dynam. Syst.*, vol. 22, pp. 273-348, 1985.

[36] Y. Liu and B. D. 0. Anderson, "Controller reduction via stable factorization and balancing," *Int. J. Contr.*, vol. 44, no. 2, pp. 507-531, 1986.

[37] E.A.Jonckheere and L.M.Silverman, "A new set of invariants for linear systems-Application to reduced order compensator design," *IEEE Trans. Automat. Contr.*, vol. 28, no. 10, pp. 953-64, 1983.

[38] W. Haddad and D. S. Bernstein, " Generalized Riccati Equations for the Full and Reduced-order Mixed-norm H^2/H^∞ Standard Problem,' Systems & Control Letters, vol.14, pp. 185-197, 1990.

[39] B. D. O. Anderson and S. Vongpanitlerd, *Network Analysis and Synthesis: A Modern Systems Theory Approach*, Prentice Hall, New Jersey, 1973.

[40] C. S. Hsu, X. Yu, H. H. Yeh, and S. S. Banda, "H^∞ Compensator Design with Minimal-order Observers," *Proceedings of American Control Conference*, 1993.

[41] B. C. Chang and A. Yousuff, "Pole Placement and the Observer-based Controller Parametrization," *IEEE Transactions on Automatic Control*, June 1990.

[42] I. Dias, A. Yousuff, and B. C. Chang, "Parametrization of Order Reducing Dynamic Feed-through Loops," *Proceedings of American Control Conference*, 1990.

[43] B. W. Choi, D. W. Gu, and I. Postlethwaite, " Low-order Stabilizing Controllers and Pole-assignment," *Proceedings of American Control Conference*, 1992.

[44] M. Aoki, "Control of Large Scale Dynamic Systems by Aggregation," *IEEE Trans. Autom. Control*, Vol.AC-13, pp. 246-253, 1968.

[45] R.E. Skelton, A. Yousuff, "Component Cost Analysis of Large Scale Systems," *Int. J. Control*, Vol.37, No.2, pp. 285-304, 1983.

[46] D.A. Wilson, "Model Reduction of Multivariable Systems," *Int. J. Control*, Vol.20,

pp. 57-64, 1974.

[47] A. Yousuff, R.E. Skelton, "An Optimal Controller Reduction by Covariance Equivalent Realizations," *IEEE Trans. Autom. Control*, Vol.AC-31, pp. 56-60, 1986.

[48] A. Yousuff, R.E. Skelton, "Controller Reduction by Component Cost Analysis," *IEEE Trans. Autom. Control*, Vol.AC-29, pp. 520-530, 1984.

[49] S.W. Greeley, D.C. Hyland, "Reduced-Order Compensation: LQG Reduction Versus Optimal Projection," *Proc. AIAA Conf. Guid. Nav. Control*, Monterey, CA, August 1987.

[50] M. Ikeda, D.D. Siljåk, "Overlapping Decompositions, Expansions, and Contractions of Dynamic Systems, " *Large Scale Systems*, Vol.1, pp. 29-38. 1980.

[51] A. Yousuff, "Application of Inclusion Principle to Mechanical Systems," *Amer. Control Conf.*, Atlanta, GA, June 15-17, 1988, pp. 1516-1520.

[52] D.S. Bernstein, "Sequential Design of Decentralized Dynamic Compensators Using the Optimal Projection Equations: An Illustrative Example Involving Interconnected Flexible Beams," *Proc. Amer, Control Conf.*, Minneapolis, MN, pp. 986-989, 1987.

[53] *NASA/DoD Control/Structure Interaction Technology*, Arlington, VA, 1986, and Denver, CO, 1987.

[54] V.B. Venkayya, V.A. Tischler, "Frequency Control and its Effect on the Dynamic Response of Flexible Structures," *AIAA Journal*, Vol. 23, pp.1768-1774, 1985.

[55] A. Hu, R.E. Skelton, T.Y. Yang, "On the Selection of the Finite Element Models for Control Design," to appear in *J. Sound and Vibration*, also in *Proc. Amer. Control Conf.*, Boston, MA, pp.1648-1656, 1985.

[56] J.L. Junkins, D.W. Rew, "A Simultaneous Structure/Controller Design Iteration Method," *in Proc. Amer. Control Conf.*, Boston, MA, pp.1642-1647, 1985.

[57] D. C. Youla, J. J. Bongiorno, H. A. Jabr, "Modern Wiener-Hopf Design of Optimal Controllers, Part II: The Multivariable Case," *IEEE Trans. Auto. Control*, Vol.AC-

21, pp.319-338, 1976.

[58] C.A. Desoer, R.W. Liu, J. Murray, and R. Saeks, "Feedback System Design: The Fractional Representation Approach", *IEEE Trans. Auto. Control*, Vol.AC-25, pp.399-412, 1980.

[59] C. N. Nett, C. A. Jacobson and M. J. Balas, "A Connection Between State-space and Doubly Coprime Fractional Representations", *IEEE Trans. Auto. Control*, Vol.AC-29, pp. 831-832, 1984.

[60] R.E. Skelton, "Cost Decomposition of Linear Systems with Application to Model Reduction," *Int. J. Control,* Vol. 32, pp.1031-1055, 1980.

[61] R.E. Skelton, P.C. Hughes, "Modal Cost Analysis of Linear Matrix Second Order Systems," *Journal of Dynamic Systems, Measurement, and Control,* Vol.102, pp.151-158, 1980.

[62] C. Commault, J.P. Guerin, "Propriétés et Structure des Systémes Agrégés Application au Probléme de la Projection Optimale," *R.A.I.R.O. Automatique/Systems Analysis and Control,* Vol.12, No.4, pp.377-390, 1978.

[63] W. Gawronski, J.N. Juang, "Model Reduction for Flexible Structures," in Control and Dynamic Systems, Vol.36, (C.T.Leondes, Ed.), Academic Press, New York, 1990.

[64] Y. Kim. J.L.J Unkins, "Measure of Controllability for Actuator Placement," *Journal of Guidance, Control, and Dynamics,* Vol.14, No.5, pp.895-902, 1991.

APPENDIX

Proof of Theorem 9.4: Write the state space parameters of Q(s) as

$$A_q = \begin{bmatrix} A_{q11} & A_{q12} \\ A_{q21} & A_{q22} \end{bmatrix}, \quad B_q = \begin{bmatrix} B_{q1} \\ B_{q2} \end{bmatrix} \tag{A-1}$$

$$C_q = \begin{bmatrix} C_{q1} & C_{q2} \end{bmatrix}$$

where

$$B_{q1} = B_j + (B_p - B_j D_p)(I + D_k D_p)^{-1} D_k \tag{A-2}$$

$$B_{q2} = B_k [\, I - D_p (I + D_k D_p)^{-1} D_k \,] \tag{A-3}$$

$$C_{q1} = C_j - (I + D_k D_p)^{-1} D_k (C_p + D_p C_j) \tag{A-4}$$

$$C_{q2} = (I + D_k D_p)^{-1} C_k \tag{A-5}$$

$$A_{q11} = (A_p - B_j C_p) + (B_p - B_j D_p) C_{q1} \tag{A-6}$$

$$A_{q12} = (B_p - B_j D_p) C_{q2} \tag{A-7}$$

$$A_{q21} = -B_{q2}(C_p + D_p C_j) \tag{A-8}$$

$$A_{q22} = A_k - B_k D_p C_{q2}. \tag{A-9}$$

Let this realization be unobservable, and let Ψ_q be the collection of eigenvectors corresponding to the unobservable eigenvalues Λ_{uo}. Then partitioning Ψ_q as shown below, the following equation must be satisfied:

$$A_q \Psi_q = A_q \begin{bmatrix} \Psi_o \\ \Psi_k \end{bmatrix} = \Psi_q \Lambda_{uo} \quad \& \quad C_q \Psi_q = 0. \tag{A-10}$$

Substituting Eqn.(A-1) in (A-10) yields

$$A_{q11}\Psi_o + A_{q12}\Psi_k = \Psi_o \Lambda_{uo} \tag{A-11}$$

$$A_{q21}\Psi_o + A_{q22}\Psi_k = \Psi_k \Lambda_{uo} \tag{A-12}$$

$$C_{q1}\Psi_o + C_{q2}\Psi_k = 0 \tag{A-13}$$

After some algebraic manipulation, Eqn.(A-11) results in (9.43b), whereas, Eqns.(A-12) and (A-13) respectively yield

$$A_k\Psi_k - B_kC_p\Psi_o = \Psi_k\Lambda_{uo} \tag{A-14}$$
$$C_k\Psi_k - D_kC_p\Psi_o = -C_j\Psi_o \tag{A-15}$$

These equations are written compactly in Eqn.(9-42a).